13. *Ergodic Theorems and the Markoff Process with a Stable Distribution.*

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(Comm. by T. TAKAGI, M.I.A., March 12, 1940.)

1. *Introduction.* A Markoff process $P(t, E)$[1] is called to have a *stable distribution* $\varphi(E)$, if there exists a completely additive non-negative set function $\varphi(E)$ (with $\varphi(\Omega)=1$) defined for all Borel set $E\subset\Omega$ such that

$$(1)\qquad \int_{\Omega}\varphi(dt)P(t, E)=\varphi(E)\quad \text{for any Borel set } E\subset\Omega .$$

For example, the Markoff process defined by a φ-measure preserving transformation $S(t)$: $P(t, E)=1$ if $S(t)\in E$, $=0$ if $S(t)\bar{\in} E$, has $\varphi(E)$ as its stable distribution; and another example thereof is given by the Markoff process with symmetric φ-density $p(t, s)$: $P(t, E)=\int_E p(t, s)\,\varphi(ds)$, $p(t, s)=p(s, t)$.

It is the purpose of the present paper to discuss such a class of Markoff processes. The same problem is also discussed by K. Yosida[2] in the preceding paper. He has proved that a mean ergodic theorem (for exact formulation see Theorem 2 below) holds for any Markoff process with stable distributions. Since this class of Markoff processes contains the case of measure preserving transformations, his result is a generalization of the mean ergodic theorem of J. v. Neumann.[3] In the present paper, we shall first prove (Theorem 1) that even an ergodic theorem of G. D. Birkhoff's type[4] is valid for such a class of Markoff processes. Indeed, we shall prove that for any *bounded* Borel measurable function $x(t)$ defined on Ω the sequence $\left\{\frac{1}{N}\sum_{n=1}^{N}x_n(t)\right\}$ $(N=1, 2, \ldots)$, where

$$(2)\qquad x_n(t)=\int_{\Omega}P^{(n)}(t, ds)x(s)\,,\qquad n=1, 2, \cdots ,$$

converges φ-almost everywhere on Ω. This result is, in essential, due to J. L. Doob.[5] We shall next show that the mean ergodic theorem

1) As for the notions concerning Markoff process, see:
S. Kakutani: Some results in the operator-theoretical treatment of the Markoff process, Proc. **15** (1939), 260–264. K. Yosida: Operator-theoretical treatment of the Markoff process, Proc. **14** (1938), 363–367, Proc. **15** (1939), 127–130.

2) K. Yosida: Markoff process with stable distribution, Proc. **16** (1940), 43–48.

3) J. v. Neumann: Proof of the quasi-ergodic hypothesis, Proc. Nat. Acad. U.S.A., **18** (1932), 70–82.

4) G. D. Birkhoff: Proof of the ergodic theorem, Proc. Nat. Acad. U.S.A., **18** (1932), 650–655.

5) J. L. Doob: Stochastic processes with an integral valued parameter, Trans. Amer. Math. Soc., **44** (1938), 87–150.