PAPERS COMMUNICATED

12. *The Markoff Process with a Stable Distribution.*

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§ 1. *Introduction and the theorems.* Let R be a space and let $B(R)$ be a completely additive family of "measurable" subsets of R. We assume that R itself belongs to $B(R)$. Let $P(x, E)$ denote the transition probability that the point $x \in R$ is transferred, by a simple Markoff process, into the set $E \in B(R)$ after the elapse of a unit-time. We naturally assume that $P(x, E)$ is completely additive for $E \in B(R)$ if x is fixed, and that $P(x, E)$ is "measurable" (with respect to $B(R)$) in x if E is fixed. Then the transition probability $P^{(n)}(x, E)$ that the point x is transferred into E after the elapse of n unit-times is given by $P^{(n)}(x, E) = \int P^{(n-1)}(x, dy) P(y, E)$[1] $\big(n=1, 2, \dots;\ P^{(1)}(x, E) = P(x, E)\big)$. We have surely

$$(1) \qquad P^{(n)}(x, E) \geqq 0\,, \quad P^{(n)}(x, R) \equiv 1 \qquad (n=1, 2, \dots)\,.$$

We now assume that there exists a non-negative set function $\varphi(E)$ which satisfies the conditions:

(2) $\varphi(E)$ is completely additive for $E \in B(R)$ and $\varphi(R) = 1$.

(3) $\left\{\begin{array}{l}\text{The space } B(R), \text{ if metrised by the distance } d(E_1, E_2) = \\ \varphi(E_1 + E_2 - E_1 \cdot E_2), \text{ is (complete and) separable.}\end{array}\right.$[2]

(4) $\int \varphi(dx) P(x, E) = \varphi(E)$ for any $E \in B(R)$.

We have, from (4), $\int \varphi(dx) P^{(n)}(x, E) = \varphi(E)$ for any n. Hence the mass distribution $\varphi(E)$ is *stable* with respect to the time. Such Markoff process (*with a stable distribution* $\varphi(E)$) is fairly general; it includes the *deterministic transition process* in the ergodic theory of the incomplessible stationary flow, originated by G. D. Birkhoff and J. von Neumann. In fact, let T be a one-to-one point transformation of R on R which maps any set $E \in B(R)$ on the set $T \cdot E \in B(R)$ in measure-preserving way: $\varphi(E) = \varphi(T \cdot E)$. Let $C_E(x)$ be the characteristic function of E and put $P(x, E) = C_E(T \cdot x)$, then it is easy to see that this $P(x, E)$ defines a Markoff process with stable distribution $\varphi(E)$. Another example is given by the Markoff process with *symmetric* φ-density: $P(x, E) = \int_E p(x, y) \varphi(dy)$, $p(x, y) \equiv p(y, x)$. Thus our general

1) The definite integral over R will be denoted by $\int \varphi(dx)$.

2) The completeness of the metrical space $B(R)$ follows from the complete additivity of $B(R)$. The separability hypothesis may be taken away, by suitably modifying the proof below. However, for the sake of brevity, I here assume it.