31. On the Theory of Almost Periodic Functions in a Group.

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The theory of almost periodic functions (a. p. f.) in a group, due originally to J. von Neumann,[1] has been simplified by W. Maak.[2] The last author starts from a modified definition of a. p. f., and obtains a shorter proof of the existence of the mean value. His proof necessitates, however, a certain combinatorial lemma, which is indeed very interesting in itself, but somewhat alien to the theory of a. p. f. We propose here another way of founding this theory, which seems to us also simple and natural.

1. We begin with some general remarks on metric spaces. An abstract space $\mathfrak{R}$ with "points" $x, y, z, \dots$ is called a metric space, if there is defined a "metric," i. e. a real-valued function $\rho(x, y)$ for $x, y \in \mathfrak{R}$ satisfying the following conditions: 1) $\rho(x, y) \geqq 0$, $\rho(x, x)=0$, 2) $\rho(x, y)=\rho(y, x)$, 3) $\rho(x, y)+\rho(y, z) \geqq \rho(x, z)$. The separation axiom: "$\rho(x, y)=0$ implies $x=y$" will not be postulated here.[3] Such spaces are topological spaces, i. e, they satisfy the first three Hausdorff axioms; it is therefore clear what are to be understood under the terms such as: open or closed sets in $\mathfrak{R}$, the continuity of a mapping of $\mathfrak{R}$ in another such space $\mathfrak{R}'$ etc. One can, moreover, speak of the equi-continuity of a family of mappings and also the uniform continuity of a mapping.[4] Theorems such as the following are evidently true thereby: If f maps $\mathfrak{R}$ continuously in $\mathfrak{R}'$, and f' maps $\mathfrak{R}'$ in $\mathfrak{R}''$ in the same way, then $f''=f'f$ maps $\mathfrak{R}$ also continuously in $\mathfrak{R}''$. If, moreover, f and f' are uniformly continuous, so is also f''. (Transitivity of continuity and uniform continuity.)

We can speak also of the diameter of a set $\mathfrak{A}$ in $\mathfrak{R}$, ε-covering, ε-net, the boundedness and the totally-boundedness of $\mathfrak{A}$. If we have to do with several metrics of a fixed space $\mathfrak{R}$, we will say also that *a metric ρ is bounded or totally bounded (t. b.)*, when the entire space $\mathfrak{R}$ has this property with respect to ρ. For two metrics ρ, ρ_1 of $\mathfrak{R}$ we will write $\rho \leqq \rho_1$, if $\rho(x, y) \leqq \rho_1(x, y)$ for all $x, y \in \mathfrak{R}$. The following lemmas are all fairly obvious:

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1) J. von Neumann: Almost periodic functions in a group I. Trans. Am. math. Soc. Vol. **36** (1934).

2) W. Maak: Eine neue Definition der fast periodischen Funktionen. Abh. math. Sem. d. Hans. Universität. **11.** Bd. (1936).

3) Such $\rho(x, y)$ is often called "quasi-metric" in opposition to the usual "metric" satisfying the separation axiom. We prefer to call ρ a "metric" in the general case, and "separated metric" when it satisfies the separation axiom.

4) In this sense $\mathfrak{R}$ is a "uniform space"; cf. André Weil; Espaces à structure uniforme. Act. sc. et ind. **551** (1937). A. Weil postulates, however, the separation axiom.