51. *Vector Lattices and Additive Set Functions.*

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§ 1. *Introduction.* One of the fundamental theorems in the theory of the integral is that of Radon-Nikodym concerning the countably additive set functions. The proof given below (§ 2) of this theorem will be shorter than the standard one in S. Saks' book[1] or the one recently published by C. Carathéodory[2]. Our proof is carried out by a *maximal method,* by making use of a lemma which is a simple modification of H. Hahn's decomposition theorem[3]. The same method is also applied (§ 3 and § 4) to give lattice-theoretic formulations of Radon-Nikodym's theorem. Of these ten years, such formulations were given more or less explicitly by many authors, F. Riesz, H. Freudenthal, Garrett Birkhoff, S. Kakutani, F. Maeda and S. Bochner-S. R. Phillips[4]. Our maximal method also makes use of the ideas of Riesz and Freudenthal, but it seems to be more direct than those of the cited authors. Thus, without appealing to Freudenthal's spectral theorem, we may obtain Kakutani's lattice-theoretic characterisation of the Banach space (L) from our result in § 3. Moreover the result in § 4 will give a simplified proof and extension of Freudenthal's spectral theorem.

§ 2. *The concrete case.* A class $\mathfrak{X}$ of sets in a space X is called *countably additive* if (i) the empty set belongs to $\mathfrak{X}$, (ii) with E its complement CE also belongs to $\mathfrak{X}$ and (iii) the sum $\bigvee_{n=1}^{\infty} E_n$ of sequence $\{E_n\}$ of sets $\in \mathfrak{X}$ belongs to $\mathfrak{X}$. A real-valued finite function $F(E)$ defined on $\mathfrak{X}$ is called *countably additive* if $F(\bigvee_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty} F(E_n)$ for any sequence $\{E_n\}$ of mutually disjoint sets $\in \mathfrak{X}$. Let $\varphi(E)$ be a *measure* on X, that is, $\varphi(E)$ be countably additive and non-negative on X. Without losing generality we assume that any subset of a set $\in \mathfrak{X}$ of φ-measure zero also belongs to $\mathfrak{X}$. A countably additive set function

1) Theory of the integral, Warsaw (1937), 36.

2) Ueber die Differentiation von Maszfunktionen, Math. Zeits., **42** (1940), 181–189. J. von Neumann also gave an interessant proof by making use of the Banach space (L_2). See his Rings of operators, III, Ann. of Math., **41** (1940), 126.

3) S. Saks: loc. cit., 32.

4) F. Riesz: Sur la décomposition des opérations linéaires, Bologna Congress, III (1928), 143–148. Sur quelques notions fondamentales dans la théorie générale des opérations linéaires, Ann. of Math., **41** (1940), 174–206. Sur la théorie ergodique des espaces abstraits, Acta Szeged, **10** (1941), 1–20, to be cited as $[R-3]$. H. Freudenthal: Teilweise geordnete Modulen, Proc. Acad. Amsterdam, **39** (1936), 641–651, to be cited as $[F-1]$. G. Birkhoff: Dependent probabilities and spaces (L), Proc. Nat. Acad., **24** (1938), 154–158. S. Kakutani: Mean ergodic theorem in abstract L-spaces, Proc., **15** (1939), 121–123. Concrete representations of abstract L-spaces and the mean ergodic theorem, Ann. of Math., **42** (1941), 523–537, to be cited as $[K-2]$. F. Maeda: Partially ordered linear spaces, J. Sci. Hirosima Univ., **10** (1940), 137–150. S. Bochner-S. R. Phillips: Additive set functions and vector lattices, Ann. of Math., **42** (1941), 316–324.