66. *On the Behaviour of a Meromorphic Function in the Neighbourhood of a Transcendental Singularity.*

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In this paper we shall prove the theorems of Beurling-Kunugui[1], Kunugui[2], and Iversen-Gross[3] using L. Ahlfors' principal theorem on covering surfaces[4].

Suppose that $f(z)$ is uniform, meromorphic in a connected domain D. Let z_0 be a point on the boundary Γ of D. We associate with z_0 three sets of values:

(1) The cluster set $S_{z_0}^{(D)}$. This is the set of all values α such that $\lim_{\nu\to\infty} f(z_\nu)=\alpha$ where z_ν $(\nu=1, 2, \dots)$ is a sequence of points tending to z_0 within D. It is obvious that $S_{z_0}^{(D)}$ is a closed set.

(2) The cluster set $S_{z_0}^{(\Gamma)}$. This is the product $\prod_{n=1}^{\infty} M_n$, where M_n denotes the closure of the sum $\sum_{0<|z'-z_0|<\frac{1}{n}} S_{z'}^{(D)}$, z' belonging to Γ. This set is also a closed set and $S_{z_0}^{(D)}$ includes $S_{z_0}^{(\Gamma)}$.

(3) The range of values $R_{z_0}^{(D)}$. A value α belongs to $R_{z_0}^{(D)}$ if, and only if, $f(z)$ takes the value α an infinity of times near z_0 inside D. It is obvious that $S_{z_0}^{(D)}$ includes $R_{z_0}^{(D)}$.

Suppose that $d(S_1, S_2)$ denotes the distance between a set S_1 and a set S_2, CS the complement of a set S with respect to the w-plane, $\bar{S}$ the closure of S, $B(S)$ the boundary of S, and $K(r)$ and $k(r)$ denote the circular disc $|z-z_0|<r$ and circumference $|z-z_0|=r$ respectively.

Lemma 1[5]. *Let $w=f(z)$ be uniform and meromorphic in a domain D, and z_0 be a point on the boundary Γ of D. Suppose that α is a value belonging to $S_{z_0}^{(D)}-S_{z_0}^{(\Gamma)}$ but not belonging to $R_{z_0}^{(D)}$. Then α is an asymptotic value of $f(z)$ at z_0 and the length of the image of its asymptotic path by $w=f(z)$ on the Riemann sphere is finite.*

Proof. We may assume that α is finite by rotating the Riemann sphere, if necessary. Since $\alpha \bar{\in} S_{z_0}^{(\Gamma)}$, $\alpha \bar{\in} R_{z_0}^{(D)}$, there exists a positive number r such that $\alpha \bar{\in} \overline{\sum_{0<|z'-z_0|\leq r} S_{z'}^{(D)}}$ where z' varies on Γ, and $f(z) \neq \alpha$ for $|z-z_0|\leqq r$ within D. Consequently there exist positive numbers

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1) K. Kunugui: Sur un théorème de MM. Seidel-Beurling, Proc. **15** (1939), 27–32.

2) K. Kunugui: Sur un problème de M. A. Beurling, Proc. **16** (1940), 361–366.

3) K. Noshiro: On the theory of the cluster sets of analytic functions, Journ. Fac. Sc. Hokkaido Imp. University. Ser. I, vol. **6** (1938), pp. 230–231.

4) L. Ahlfors: Zur Theorie der Überlagerungsflächen, Acta Math., Bd. **65** (1935), or R. Nevanlinna: Eindeutige analytischen Funktionen, (1936), pp. 312–345.

5) Noshiro: loc. cit. theorem 1. p. 221. He proved that α is an asymtotic value of $f(z)$ at z_0 under the same hypothesis.