90. On the Behaviour of an Inverse Function of a Meromorphic Function at its Transcendental Singular Point.

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Let $w=f(z)$ be a meromorphic function for $|z|<\infty$ with a transcendental singularity at $z=\infty$ and its inverse function $z=\varphi(w)$ have a transcendental singularity ω with $w=0$ as its projection on the w-plane. Denote Δ the set of values taken by $z=\varphi(w)$ which defines the ρ-neighbourhood of an accessible boundary point ω of the Riemann surface F of $z=\varphi(w)$. Δ is a domain on the z-plane bounded, in general, by an enumerable infinity of analytic curves. $|w|=|f(z)|<\rho$ in Δ and $|w|=\rho$ on the boundary of Δ. Let z_0 be a point in Δ. The common part of Δ and $|z-z_0|<r$ consists of a certain number of connected domains. Let Δ_r be one of such domains which contains z_0. The boundary of Δ_r consists of curves of three types: $\{a_i^{(r)}\}$, $\{b_i^{(r)}\}$, $\{c_i^{(r)}\}$, where $a_1^{(r)}, a_2^{(r)}, \dots, a_{n_r}^{(r)}$ are circular arcs on $|z-z_0|=r$, $b_1^{(r)}$, $b_2^{(r)}, \dots, b_{m_r}^{(r)}$ are the parts of the boundary of Δ which meet $|z-z_0|=r$, and $c_1^{(r)}, c_2^{(r)} \dots, c_{p_r}^{(r)}$ are the closed curves which are the boundaries of holes in Δ_r. We put

$$\Lambda(r)=p_r=\text{number of holes in } \Delta_r\,. \tag{1}$$

Let F_r correspond to Δ_r on F and $A(r)$ be the area of F_r and put

$$S(r)=\frac{A(r)}{\pi\rho^2}\,. \tag{2}$$

$A(r)$ is an increasing function of r and is continuous except at most an enemerable infinity of points $\{a_i\}$, where $A(a_i-0)=A(a_i)$. Let $A_i^{(r)}$, $B_i^{(r)}$, $C_i^{(r)}$ correspond to $a_i^{(r)}$, $b_i^{(r)}$, $c_i^{(r)}$ on F_r and $L_i^{(r)}$ be the length of $A_i^{(r)}$ and put $L(r)=L_1^{(r)}+L_2^{(r)}+\cdots+L_{n_r}^{(r)}$. We will show that

$$\lim_{r\to\infty} \Lambda(r)=\infty \tag{3}$$

and there exists a sequence $\{r_n\}$ tending to infinity, such that

$$\frac{L(r)}{S(r)}\to 0\,, \qquad \text{when} \quad r=r_n\to\infty\,. \tag{4}$$

In the following we consider only such $r=r_n$.

We will prove (3) and (4) by modifying slightly Mr. Noshiro's[1)]

1) K. Noshiro: On the singularities of analytic functions. Japanese Journal of Mathematics, **17** (1940), 37–96.