11. An Abstract Integral, V.

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Introduction. S. Banach has introduced an integral which has no convergence properties, and which is defined for all bounded functions in $(0, 1)$. Evidently this class does not contain the class (L) of Lebesgue integrable functions. Since Banach integral is the integral without convergence properties, it will be desirable to define the Banach integral so that the class (B) of Banach integrable functions contains the class (L) and if Banach integral is pressed to have convergence properties of Lebesgue, then (B) reduces to (L). This is possible by the Jessen-Khintchine theorem.

In the case of abstract-integral, it is desirable to define such integral. For this purpose, we have introduced the abstract Banach integral for which above relation holds for the abstract-Lebesgue integrals in the third and fourth papers. This is given in § 1. In § 2 we define the second Banach integral such that above relation holds for abstract Riemann integral, § 3 contains a certain uniqueness theorem of above two integrals.

§ 4 contains that above consideration can be extended to the case where the value of the integral lies in a semi-vector lattice instead of real number field.

§ 1. Let E be a partially ordered linear space whose elements are denoted by $x, y, \dots$, and M be a set of elements $\alpha, \beta, \dots$. Now we shall consider a set of operations $T^\alpha x (\alpha \in M)$ which transforms E into the space of real numbers, and satisfies the following conditions.

(1.1) For every elements α, β of M and x, y of E, there exists a γ of M such that $T^\gamma(x+y) \leqq T^\alpha x + T^\beta y$.

(1.2) $\lambda T^\alpha x = T^\alpha(\lambda x)$, for any real number λ.

(1.3) If $x \leqq 0$, then $T^\alpha x \leqq 0$.

(1.4) For any element α of M, there exists an element e of M such that $T^\alpha e = 1$.

If we put

(1.5) $p(x) \equiv \text{g. l. b.} (T^\alpha x\,;\, \alpha \in M)$[1],

then we have

(1.6) $p(x+y) \leqq p(x) + p(y)$ for every x and y in E.

(1.7) $p(tx) = tp(x)$ for $t \geqq 0$.

Proof. For every $\varepsilon > 0$ and every x, y in E, we can find α and β in M such that

1) g. l. b. $(T^\alpha x\,;\, \alpha \in M)$ means the greatest lower bound of $T^\alpha x$ when α runs over M. For the least upper bound we use the similar notation.