PAPERS COMMUNICATED

27. Notes on Banach Space (I): Differentiation of Abstract Functions.

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The object of this paper is to generalize the key theorem due to Pettis[1], from which almost all theorems concerning differentiation of the Banach space-valued functions of bounded variation are derived.

1. We will begin by some known definitions.

Let $\mathfrak{X}$ be a Banach space[2].

[1.1][1] If (γ_n) is a sequence in $\bar{\mathfrak{X}}$ (conjugate space of $\mathfrak{X}$) and $\mathfrak{Y}$ is a subset of $\mathfrak{X}$, then (γ_n) is called to have property $N(\mathfrak{Y})$ provided that $\|\gamma_n\| \leqq 1$ $(n=1,2,\ldots)$ and $\|y\| = \limsup_n |\gamma_n(y)|$ for every $y \in \mathfrak{Y}$. In this case we write symbolically $(\gamma_n) \in N(\mathfrak{Y})$.

[1.2][1] If X_R is an interval function on a Euclidean cube R_0 to $\mathfrak{X}$, then $\big(\pi_n \equiv (R_n, 1, \ldots, R_n, k_n)\big)$ is called a X_R-maximal sequence provided that π_n is a partition of R_0 and $\sum_{i=1}^{k_n} \|X_{R_{n,i}}\|$ tends to the total variation of X_R on R_0 as $n \to \infty$.

[1.3][1] If $\pi_n = (R_{n,1}, \ldots, R_{n,k_n})$ $(n=1,2,\ldots)$ is a sequence of partitions of R_0, then the linear closure of the set $(X_{R_{n,i}}; i=1,2,\ldots,k_n; n=1,2,\ldots)$ is called the X_R-span of (π_n).

[1.4][3] If x_s is a point function on R_0 to $\mathfrak{X}$, then x_s is called to be restricted provided that 1° x_s is bounded, 2° there exists a positive number M such that for any $\varepsilon > 0$ there corresponds a sequence of disjoint measurable sets $(e_1, e_2, \ldots)$ with $|e_i| < \varepsilon$, $\sum_{i=1}^{\infty} |e_i| = |R_0|$ and $\|\sum (x_{\xi_{i_k}} - x_{\eta_{i_k}})\| < R$ for any subset (i_k) and ξ_{i_k} and η_{i_k} in e_{i_k}.

[1.5][3] x_s is called to be measurable if there exists a sequence of restricted functions which tends to x_s. Symbolically we write $x_s \in M$.

[1.6] If x_s is integrable in the Birkhoff sense, then we write $x_s \in L$.

Concerning differentiation, we define

[1.7][1] If $\mathfrak{Z}$ is a subset of $\bar{\mathfrak{X}}$, then X_R is said to be $\mathfrak{Z}$-pseudo-differentiable to x_s almost everywhere provided that $\zeta(X_R)$ is differentiable (in the ordinary sense) to $\zeta(x_s)$ almost everywhere for all ζ in $\mathfrak{Z}$.

[1.8] X_R is (strongly) differentiable to x_s almost everywhere if for almost all s in R_0 $X_R/|I|$ tends to x_s when I is a cube containing s and $|I|$ tends to zero.

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1) Pettis, Duke Math. Journ., **5** (1938).
2) [ ] denotes definition and ( ) theorem.
3) Jeffery, Duke Math. Journ.. **9** (1941).