36. On a Theorem of F. and M. Riesz.

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1. Let D be a domain on the w-plane, bounded by a rectifiable curve Γ and we map D conformally on $|z|<1$, then F. and M. Riesz[1] proved that a null set on $|z|=1$ corresponds to a null set on Γ and a null set on Γ corresponds to a null set on $|z|=1$, where a set is called a null set, if its measure is zero. We will prove an analogous theorem, when D is a domain on a minimal surface, bounded by a rectifiable curve.

Let Γ be a rectifiable curve in an m-dimensional space, then it is proved by Radó, Douglas and Courant that there exists a minimal surface S through Γ.

Let S be defined by a vector $\mathfrak{x}=\mathfrak{x}(z)=\big(x_1(z), \dots, x_m(z)\big)$ $(z=u+iv=re^{i\theta})$, where the components $x_k(z)$ $(k=1,\dots,m)$ are continuous in $|z|\leqq 1$ and harmonic in $|z|<1$ and $\mathfrak{x}=\mathfrak{x}(e^{i\theta})$ maps $|z|=1$ continuously and monotonically on Γ and if we put

$$E=\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial u}\right)^2, \quad F=\sum_{k=1}^{m}\frac{\partial x_k}{\partial u}\cdot\frac{\partial x_k}{\partial v}, \quad G=\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial v}\right)^2,$$

then

$$E=G, \quad F=0 \quad \text{in } |z|<1. \tag{1}$$

Let ds be the line element on S, then

$$ds^2=\sum_{k=1}^{m} dx_k^2=E(du^2+dv^2)=E(dr^2+r^2d\theta^2), \tag{2}$$

so that

$$E=E(z)=\frac{1}{r^2}\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial \theta}\right)^2.$$

Put $x_k=\Re\big(f_k(z)\big)$, where $f_k(z)$ are regular in $|z|<1$, then

$$\begin{aligned}E&=\frac{1}{2}(E+G)=\frac{1}{2}\sum_{k=1}^{m}\left(\left(\frac{\partial x_k}{\partial u}\right)^2+\left(\frac{\partial x_k}{\partial v}\right)^2\right)\\&=\frac{1}{2}\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial u}+i\frac{\partial x_k}{\partial v}\right)\left(\frac{\partial x_k}{\partial u}-i\frac{\partial x_k}{\partial v}\right)=\frac{1}{2}\sum_{k=1}^{m}|f_k'(z)|^2.\end{aligned} \tag{3}$$

We will prove the following theorem.

Theorem I. Let S be a minimal surface in an m-dimensional space, bounded by a rectifiable curve Γ and $\mathfrak{x}=\mathfrak{x}(z)$ map S on $|z|\leqq 1$, then a null set on $|z|=1$ corresponds to a null set on Γ and a null set on Γ corresponds to a null set on $|z|=1$.

1) F. and M. Riesz: Über die Randwerte einer analytischen Funktion. Quatrième congres des mathématiciens scandinaves à Stockholm, 1916.