39. *Notes on Infinite Product Measure Spaces, II.*

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1. Let $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$ be a family of measure spaces satisfying $m^\gamma(\Omega^\gamma)=1$ for each $\gamma \in \Gamma$. In the first paper[1] we have shown that there exists an independent infinite product measure space $(\Omega^*, \mathfrak{B}^*, m^*) = P \otimes_{\gamma \in \Gamma} (\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma)$ of these measure spaces. We have proved the existence of the measure $m^*(B^*)$ on the Borel field $\mathfrak{B}^*$ without assuming any topology on the spaces Ω^γ and without using any notion of topology in the proof. In the present paper, however, we shall discuss the case when each space Ω^γ, and hence the infinite product space Ω^*, is a compact Hausdorff space. These assumptions on the spaces Ω^γ will make it possible to discuss the properties of the independent product measure space in more detail. We shall see that there usually exist two infinite independent product measure spaces $(\Omega^*, \mathfrak{B}^*, m^*)$ and $(\Omega^*, \mathfrak{B}^{**}, m^{**})$ defined on the same product space Ω^*, of which the latter is an extension of the former and which usually do not coincide. We shall discuss the conditions under which the completions of these measure spaces coincide, and the results thus obtained will find some applications to the theory of Haar measures on non-separable locally compact topological groups. It is also to be noted that the idea of defining the measure for every open, and hence for every Borel, subset of the product space Ω^* has important consequences in the theory of continuous stochastic processes and of Brownian motions. We shall, however, not enter into these problems in this note, and the discussions of details of the applications are left to another occasion.

2. We begin with preliminary remarks. Let Ω be a compact Hausdorff space, and let $\mathfrak{B}_\Omega$ be the Borel field of subsets B of Ω which is generated by the family $\mathfrak{O}_\Omega$ of all open subsets O of Ω. A subset B of Ω which belongs to $\mathfrak{B}_\Omega$ is called a *Borel subset* of Ω. A countably additive measure $m(B)$ defined on $\mathfrak{B}_\Omega$ $\left(\text{satisfying } m(\Omega) < \infty\right)$ is *regular* if there exists for any $B \in \mathfrak{B}_\Omega$ and for any $\varepsilon > 0$ an $O \in \mathfrak{O}_\Omega$ such that $B \subseteqq O$ and $m(O) < m(B) + \varepsilon$.

Let us further denote by $C(\Omega)$ the Banach space of all bounded real-valued continuous functions $x(\omega)$ defined on Ω with $\|x\| = \sup_{\omega \in \Omega} |x(\omega)|$ as its norm. $C(\Omega)$ is at the same time a real normed ring with respect to the ordinary operation of product, and may also be considered as a Banach lattice if we put

(1) $\quad x \geqq y$ if and only if $x(\omega) \geqq y(\omega)$ for all $\omega \in \Omega$.

In fact, $C(\Omega)$ is a so-called (M)-space with respect to this partial

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1) S. Kakutani, Notes on Infinite Product Measure Spaces, I, Proc. **19** (1943), 148.