PAPERS COMMUNICATED

36. *Notes on Banach Space (VI): Abstract Integrals and Linear Operations.*

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The object of this paper is to give general representation theorems of linear operations from a Banach space to another where one is a concrete Banach space. In this direction there are many results due to Gelfand[1], Kantorovitch-Vulich[2], Kantorovitch[3] and Phillips[4], etc. In §§ 3–4 their results are all generalized and simplified. Our problem is closely connected with the integration theory. In § 2 we define abstract integrals using idea of von Neumann and Dunford[5]. These integrals are used in the representation theorems. In § 1 we state notations and theorems which are used throughout the paper.

1. *Notations and theorems due to Dunford.*

Let X be a Banach space of numerical functions $\phi(t)$, where t ranges over an abstract set T such that

1°. if $\phi_1(t)+\phi_2(t)=\phi(t)$ for all t in T, then $\phi_1+\phi_2=\phi$,

2°. if $c\phi_1(t)=\phi(t)$ for a constant c and for all t in T, then $c\phi_1=\phi$,

3°. if $\phi_n\to\phi$ and $\phi_n(t)\to\phi^*(t)$ for all t in T, then $\phi=\phi^*$,

4°. if $\phi_n \rightharpoonup \phi$, then $\phi_n(t)\to\phi(t)$ for all t in T.

Examples of such X are c, l^p $(1\leqq p\leqq\infty)$, C, B, AC and V^p $(1\leqq p\leqq\infty)$ where V^1 denotes the space of all completely additive set functions on an abstract set. In the following X denotes always such Banach space. But in §§ 1–2 we need not the condition 4°. Since L^p $(1\leqq p\leqq\infty)$ satisfies conditions 1°–3°, the results in §§ 1–2 are applicable to such spaces.

Let Y be an arbitrary Banach space and Γ a closed linear manifold in $\bar{Y}$. The linear space $\mathfrak{X}\equiv X(Y,\Gamma)$ is, by definition, the space of all abstract functions $y(.)=y(t)$ on T to Y such that $\gamma f(.)$ lies in X for every γ in Γ. $y(.)$ and $\gamma y(.)$ represent points in the function space from T to Y and X, respectively.

The following theorems are due to Dunford. We prove them for the sake of completeness.

(1.1) *If $y(.)\in X(Y,\Gamma)$, then $\gamma y(.)$ is a linear operation on Γ to X. In other words there exists a smallest non-negative number $\|y(.)\|$ such that*

$$\|\gamma y(.)\|_X\leqq\|y(.)\|\cdot\|\gamma\| \qquad (\gamma\in\Gamma).$$

1) Gelfand, Recueil Math., Moscou, **4** (1938), pp. 235–284.
2) Kantorovitch-Vulich, Comp. Math., **5** (1937), pp. 119–165.
3) Kantorovitch, Recueil Math., Moscou, **7** (1940), pp. 207–279.
4) Phillips, Trans. Amer. Math., Soc., **44** (1940), pp. 516–541.
5) Dunford, Ibidem, **44** (1936), pp. 305–356.