48. *Relation between the Measures* $\wedge_a(X)$ *and* $m^*(X)$.

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A function of set $\Gamma(X)$, which is defined and non-negative for all sets of a metrical space, will be called an outer measure in the sense of Carathéodory or simply Carathéodory measure, if it subject the following conditions:

(C- I) $\Gamma(X) \leqq \Gamma(Y)$ whenever a set Y includes a set X,

(C- II) $\Gamma(\sum_{n=1}^{\infty} X_n) \leqq \sum_{n=1}^{\infty} \Gamma(X_n)$ for every sequence $\{X_n\}_{n=1,2,\dots}$ of sets,

(C-III) $\Gamma(X+Y)=\Gamma(X)+\Gamma(Y)$ whenever the distance $\rho(X, Y)>0$.

Now in any separable metrical space, we can define, as the following manner, a function of set which is one of Carathéodory measures.

Let a be an arbitrary positive number. Given a set in the space, we shall denote, for each positive number ε, by $\wedge_a^{(\varepsilon)}(X)$ the greatest lower bound of the sums $\sum_{n=1}^{\infty}[d(X_n)]^a$, for which $(X_n)_{n=1,2,\dots}$ is an arbitrary partition of the set X into a sequence of sets whose diameters $d(X_n)$ are less than ε.

When ε tends to 0, the number $\wedge_a^{(\varepsilon)}(X)$ tends, in a monotone non-decreasing manner, to a unique limit (finite or infinite), which we shall denote by $\wedge_a(X)$. The function $\wedge_a(X)$ of set thus defined is an outer measure in the sense of Carathéodory.

For, when $\varepsilon>0$, we clearly have,

(I) $\wedge_a^{(\varepsilon)}(X) \leqq \wedge_a^{(\varepsilon)}(Y)$ whenever $X \subset Y$,

(II) $\wedge_a^{(\varepsilon)}(\sum_{n=1}^{\infty} X_n) \leqq \sum_{n=1}^{\infty} \wedge_a^{(\varepsilon)}(X_n)$ for any sequence of sets,

(III) $\wedge_a^{(\varepsilon)}(X+Y)=\wedge_a^{(\varepsilon)}(X)+\wedge_a^{(\varepsilon)}(Y)$ if $\rho(X, Y)>\varepsilon$.

Making here $\varepsilon \to 0$, (I), (II) and (III) become respectively the three conditions (C–I), (C–II) and (C–III). (vide; Saks: Theory of the integral, § 8, Chap.-II.)

In particular, we shall study the relation of $\wedge_2(X)$ thus defined and Lebesgue outer measure $m^*(X)$ in the two-dimensional Euclidean space.

First, we shall show two necessary lemmas.

Lemma I. Among the class of all sets whose diameters do not exceed a given constant, the set having the largest Lebesgue outer measure is the circle. In the other words $d(E)^2\pi/4 \geqq m^*(E)$ for each bounded set E. (vide; Bonnesen-Fenchel; Theorie d. Konvexen Körper, S. 76 u. 107.)

Lemma II. Let $\{I_n\}_{n=1,2,\dots}$ be an arbitrary sequence of the closed circles which covers a set X. Then the greatest lower bound of the sums $\sum_{n=1}^{\infty} m^*(I_n)$ coincides with Lebesgue outer measure of X.