45. *Banach Limits and the Čech Compactification of a Countable Discrete Set.*

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§ 1. Let (m) be the Banach space of all bounded sequences of real numbers $x=\{x_n \mid n=1, 2, \dots\}$ with $\|x\|=\sup_n |x_n|$ as its norm, and let $\bar{\Omega}$ be the Čech compactification (see § 2) of a countable discrete set Ω. The purpose of this paper is to give further investigations into the relations between Banach limits defined on (m) and countably additive measures defined on $\bar{\Omega}$, which were previously discussed by one of the present authors[1]. By means of this compactification $\bar{\Omega}$, we shall first obtain a general form of bounded linear functionals defined on (m) (Theorem 2). This may be considered as a generalization of a result due to L. W. Cohen and N. Dunford[2]; and a result of G. Fichtenholz and L. Kantorovitch[3] concerning the cardinal number of the conjugate space of (m) may be obtained from this easily (Theorem 3).

It will then be shown that a Banach limit corresponds to a countably additive measure $m(\bar{B})$ defined for all Borel subsets $\bar{B}$ of $\bar{\Omega}$ vanishing identically on Ω. Thus the problems of Banach limits are reduced to the problems of measures on the compact (=bicompact) Hausdorff space $\Omega'=\bar{\Omega}-\Omega$. We shall prove that there exists a family of mutually disjoint non-empty open-and-closed subsets of Ω' with the cardinal number $\mathfrak{c}$ (Theorem 6). As an application of this result we shall finally show that, given a sequence $\{f_n(x) \mid n=1, 2, \dots\}$ of bounded linear functionals defined on a closed linear subspace X_0 of a Banach space X, it is not always possible to extend each $f_n(x)$ to a bounded linear functional $F_n(x)$ defined on X in such a way that the new sequence $\{F_n(x) \mid n=1, 2, \dots\}$ is weakly convergent on the whole space X. In fact, if we take $X=(m)$, $X=(c)$= the subspace of (m) consisting of all convergent sequences $x=\{x_n \mid n=1, 2, \dots\}$, and $f_n(x)=x_n-x_{n+1}$, $n=1, 2, \dots$, then this is a required example (see § 7). As was kindly communicated to the autnors by S. Izumi, this fact was already noticed by R. S. Phillips[4]; but our method of proof is entirely different from his and may be observed with some interest.

§ 2. We begin with preliminary remarks. First we note that (m)

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1) S. Kakutani, Concrete representation of an abstract (M)-space and the characterization of the space of continuous functions, Annals of Math., **42** (1941).

2) L. W. Cohen and N. Dunford, Transformation in sequence spaces, Duke Math. Journ., **3** (1939), 689–701.

3) G. Fichtenholz and L. Kantorovitch, Sur les opérations dans l'espace des fonctions bornées, Studia Math., **5** (1934), 69–98.

4) R. S. Phillips, On linear transformations, Trans. Amer. Math. Soc., **44** (1939), 516–541.