74. *On Cardinal Numbers Related with a Compact Abelian Group.*

By Shizuo KAKUTANI.

Mathematical Institute, Osaka Imperial University.

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§1. Throughout the present paper we use the following notation:

(1) $\mathfrak{p}(A)$ = the cardinal number of a set A.

Let G be a compact abelian group containing an infinite number of elements, and let us put

(2) $\mathfrak{v}(G)$ = the smallest cardinal number $\mathfrak{p}(\Gamma)$ of a system $\mathfrak{V}(0) = \{V_\gamma(0) \mid \gamma \in \Gamma\}$ of open neighborhoods $V_\gamma(0)$ of the zero element 0 of G which defines[1] the topology of G at 0,

(3) $\mathfrak{o}(G)$ = the smallest cardinal number $\mathfrak{p}(\Gamma)$ of a system $\mathfrak{O} = \{O_\gamma \mid \gamma \in \Gamma\}$ of open subsets O_γ of G which defines[2] the topology of G,

(4) $\mathfrak{d}(G)$ = the smallest cardinal number $\mathfrak{p}(D)$ of a subset D of G which is everywhere dense in G.

The purpose of the present paper is to evaluate the cardinal numbers $\mathfrak{p}(G)$, $\mathfrak{v}(G)$, $\mathfrak{o}(G)$ and $\mathfrak{d}(G)$ in terms of the cardinal number $\mathfrak{m} = \mathfrak{p}(G^*)$ of the discrete character group G^* of G. The main results may be stated as follows:

Theorem 1. $\mathfrak{p}(G) = 2^{\mathfrak{m}}$.

Theorem 2. $\mathfrak{v}(G) = \mathfrak{o}(G) = \mathfrak{m}$.

Theorem 3. $\mathfrak{d}(G) = \mathfrak{n}$, *where* $\mathfrak{n}$ *is the smallest cardinal number which satisfies* $2^{\mathfrak{n}} \geqq \mathfrak{m}$.

Theorem 1 is a generalization of the fact that a compact abelian group containing an infinite number of elements has always a cardinal number $\geqq \mathfrak{c}$, and that there is no compact abelian group whose cardinal number is exactly $\aleph_0$. Further, assuming the generalized continuum hypothesis: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, it follows from Theorem 1 that there is no compact abelian group whose cardinal number is exactly $\aleph_\alpha$ if α is a limit ordinal. Theorem 2 implies as a special case that a compact abelian group G is separable[3] (and hence metrisable) if and only if the discrete character group G^* of G is countable, and if and only if

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1) A system $\mathfrak{V}(a) = \{V_\gamma(a) \mid \gamma \in \Gamma\}$ of neighborhoods $V_\gamma(a)$ of a point a of a topological space Ω defines the topology of Ω at a if, for any neighborhood $V(a)$ of a in Ω, there exists a $\gamma \in \Gamma$ such that $V_\gamma(a) \leqq V(a)$.

2) A system $\mathfrak{O} = \{O_\gamma \mid \gamma \in \Gamma\}$ of open subsets O_γ of a topological space Ω defines the topology of Ω if, for any $a \in \Omega$ and for any neighborhood $V(a)$ of a in Ω, there exists a $\gamma \in \Gamma$ such that $a \in O_\gamma \leqq V(a)$.

3) A topological space Ω is separable (=satisfies the second countability axiom of Hausdorff) if there exists a countable family $\mathfrak{O} = \{O_n \mid n = 1, 2, \ldots\}$ of open subsets O_n of Ω which defines the topology of Ω.