73. Normed Ring of a Locally Compact Abelian Group.

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§ 1. Let G be a locally compact (not necessarily separable) abelian group, and let $L^2(G)$ be the generalized Hilbert space of all complex-valued functions $x(g)$ which are defined, measurable and square integrable on G with respect to a Haar measure of G (with a certain fixed normalization) having

$$(1) \qquad \|x\| = \left(\int_G |x(g)|^2 dg\right)^{\frac{1}{2}}$$

as its norm. Let further $\mathfrak{B}(G)$ be the ring of all bounded linear transformations B which map $L^2(G)$ into itself. Then $\mathfrak{B}(G)$ is a (non-commutative) normed ring[1] with respect to the norm

$$(2) \qquad |||B||| = \sup_{|x|\leq 1} \|B(x)\| .$$

For each $a \in G$, let us denote by U_a a unitary transformation of $L^2(G)$ onto itself which is defined by

$$(3) \qquad U_a(x) = x_a , \qquad x_a(g) = x(g-a) .$$

Then $\mathfrak{U}(G) = \{U_a \mid a \in G\}$ is a group of unitary transformations which is algebraically isomorphic with G. Let further $\mathfrak{A}(G)$ be an algebraic subring of $\mathfrak{B}(G)$ which is generated by $\mathfrak{U}(G)$, i. e. a subring of $\mathfrak{B}(G)$ consisting of all $A \in \mathfrak{B}(G)$ of the form:

$$(4) \qquad A = \sum_{p=1}^{k} \alpha_p U_{a_p} ,$$

where $\{a_1, \dots, a_k\} \subseteqq G$ and $\{\alpha_1, \dots, \alpha_k\}$ is an arbitrary finite system of complex numbers. Let further $\mathfrak{R}(G)$ be the closure of $\mathfrak{A}(G)$ in $\mathfrak{B}(G)$, i. e. a subring of $\mathfrak{B}(G)$ consisting of all $B \in \mathfrak{B}(G)$ such that for any $\varepsilon > 0$ there exists an $A \in \mathfrak{A}(G)$ satisfying $|||B-A||| < \varepsilon$.

The purpose of this paper is to determine a general form of maximal ideals of $\mathfrak{R}(G)$. It will be shown that there exists a one-to-one correspondence between the family $\mathfrak{M}(G)$ of all maximal ideals M of $\mathfrak{R}(G)$ and the family $\mathfrak{X}(G)$ of all algebraic (=not necessarily continuous) characters[2] $\chi(a)$ defined on G. This correspondence is even

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1) I. Gelfand, Normierte Ringe, Recueil Math., **9** (1941), 3-25.

2) Under a *character* of a locally compact abelian group G, we understand a continuous representation of G by the additive group of real numbers mod. 1. Sometimes it is also necessary to consider representations of G which are not necessarily continuous. In order to distinguish these cases, we usually say *continuous characters* and *algebraic characters* of G.