# 72. Normed Rings and Spectral Theorems.

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1. *Introduction.* The purpose of the present note is to show that the simultaneous resolution of the identity for commutative ring of normal operators may easily be deduced from the theory of normed ring[1] by making use of two elementary lemmas in operator theory and of Baire's category theorem. As in the preceding notes[2], our treatment is rather algebraical and integration-free.

2. *Preliminaries.* Let $\boldsymbol{B}$ denote the totality of bounded linear operators in a general euclid space $\mathfrak{H}$. Let a subset $\boldsymbol{A}$ of $\boldsymbol{B}$ satisfy

(1) the commutativity: $TS=ST$ for $T, S \in \boldsymbol{A}$

and

(2) the conjugated condition: if $T \in \boldsymbol{A}$, then its adjoint $T^*$ also $\in \boldsymbol{A}$.

Let $\boldsymbol{A}'$ denote the totality of operators $\in \boldsymbol{B}$ that commute with every operator of $\boldsymbol{A}$, then it is easy to see that $\boldsymbol{R}=\boldsymbol{A}''=(\boldsymbol{A}')'$ is a commutative, conjugated ring with unit and with complex multipliers. $\boldsymbol{R}$ is a normed ring[3] by the norm $\|T\|=\sup_{|f|\leq 1}\|T\cdot f\|$. Moreover it is easy to see that $\boldsymbol{R}$ is closed in the sense of the *strong convergence*.

$$(3)\quad \begin{cases} \text{let a sequence } \{T_n\} \subseteqq \boldsymbol{R} \text{ be such that } \operatorname{strong}\lim_{n\to\infty} T_n \text{ exist}^{4)}, \\ \quad\text{then the operator } T=\operatorname{strong}\lim_{n\to\infty} T_n \text{ also belongs to } \boldsymbol{R}. \end{cases}$$

*Lemma 1*[5]. Let $H \in \boldsymbol{B}$ be hermitian, viz. $H=H^*$, then

$$(4)\quad \|H\|=\sup_{|f|\leq 1}\|H\cdot f\|=\sup_{|f|\leq 1}|(H\cdot f, f)|.$$

*Lemma 2*[6]. Let $T \in \boldsymbol{B}$ and let $I$ denote the identity operator, then $(I+TT^*)$ admits the inverse $(I+TT^*)^{-1} \in \boldsymbol{B}$. It is easy to see that $(I+TT^*)^{-1} \in \boldsymbol{R}$ in case $T \in \boldsymbol{R}$.

From lemma 1 we obtain

$$(5)\quad \|T\|=\|T^*\|,\ \|T^2\|=\|T\|^2 \quad \text{for every} \quad T \in \boldsymbol{R}.$$

*Proof.* We have, by (4), since $H=TT^*$ is hermitian $\left(=(TT^*)^*\right)$, $\|T\|^2=\sup_{|f|\leq 1}(T\cdot f, T\cdot f)=\sup_{|f|\leq 1}|(T^*T\cdot f, f)|=\|T^*T\|=\|TT^*\|=\|H\|$. Thus $\|T\|^2=\|T^*\|^2$ and $\|T^2\|^2=\|T^{*2}T^2\|=\|(T^*T)^2\|$ by the commutativity of $\boldsymbol{R}$.

1) I. Gelfand: Rec. Math., **9** (1941).
2) K. Yosida and T. Nakayama: Proc., **18** (1942) and **19** (1943).
3) $\boldsymbol{R}$ is a Banach space by the norm $\|T\|$ and satisfies $\|TS\|\leqq\|T\|\cdot\|S\|$.
4) $\operatorname{strong}\lim_{n\to\infty} T_n=T$ means that $\lim_{n\to\infty} T_n f=T f$ strongly for every $f \in \mathfrak{H}$.
5) See, for example, F. J. Murray's book: Princeton (1941), 41.
6) Murray s book, 42.