93. On Analytic Functions in Abstract Spaces.

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§ 1. The purpose of the present paper is to extend some of the theorems of H. Cartan[1] on functions of several complex variables to the case of functions whose domain and range both lie in complex Banach spaces.[*]

Let E and E' be two complex Banach spaces, and let $x'=f(x)$ be an E'-valued function defined on a certain neighborhood $V(x_0)$ of a point $x_0 \in E$. $x'=f(x)$ is said to admit a *variation* or a *Gateaux differential* at $x=x_0$ if

$$(1) \qquad \lim_{\alpha\to 0} \frac{f(x_0+\alpha y)-f(x_0)}{\alpha}$$

exists strongly for any $y \in E$ (α is a complex number).

An E'-valued function $x'=f(x)$ defined on a domain D of E is *analytic* in D if it is strongly continuous on D and if it admits a Gateaux differential at every point of D. It is clear that, in case both E and E' are the field of complex numbers, this definition coincides with the usual definition of a complex-valued analytic function of a single complex variable. Further, if E is the field of complex numbers while E' is an arbitrary complex Banach space, then our definition coincides with that of a Banach-space-valued analytic function of a single complex variable given by E. Hille and N. Dunford.[2]

An E'-valued function $x'=p(x)$ defined on E is a *polynomial of degree* n if the following conditions are satisfied: 1) $p(x)$ is strongly continuous at each point of E, 2) for each x and y in E, and for any complex number α, $p(x+\alpha y)$ can be expressed as

$$(2) \qquad p(x+\alpha y)=\sum_{k=1}^{n}\alpha^k p_k(x, y),$$

where $p_k(x, y)$ are arbitrary E'-valued functions of two variables x and y, 3) $p_n(x, y)\neq 0$ for some x and y. If, in addition to these, $p(\alpha x)=\alpha^n p(x)$, then the function $p(x)$ is called a *homogeneous polynomial of degree* n. It is clear that an E'-valued polynomial defined on E is analytic on E.

We shall state a theorem of A. E. Taylor[3] which we shall need in the following discussions:

Let E and E' be two complex Banach spaces. If an E'-valued

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1) H. Cartan, Sur les groupes des transformations analytiques, Actualités, Paris, 1938.

2) Cf. E. Hille, Semi-group of linear transformations, Annals of Math., **40** (1939).

3) A. E. Taylor, On the properties of analytic functions in abstract spaces, Math. Annalen, **115** (1938).