

86. Notes on Banach Space (VII): Compactness of Function Spaces.

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Recently Prof. Izumi¹⁾ has derived from his key theorem the following theorem.

A set \mathfrak{F} in E where E is (C) , $(L^p, \infty > p \geq 1)$, etc. is compact when and only when

- 1° $\|f(x)\| \leq M$ for all $f(x) \in \mathfrak{F}$,
- 2° $\lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_0^\delta f(x+t)dt = f(x)$ uniformly in \mathfrak{F} .

These conditions are of the Kolmogoroff²⁾-Tulajkov³⁾ type. But there are conditions of the Arzèla-M. Riesz⁴⁾ type. In the present note the author establishes an abstract theorem of the latter type.

Theorem. Let E be a Banach space satisfying $\lim_{t \rightarrow 0} \|f(x+t) - f(x)\| = 0$. If 1° and 2° are compactness conditions of a set \mathfrak{F} in E , then they are equivalent to the following

- 1° $\|f(x)\| \leq M$ for all $f(x) \in \mathfrak{F}$,
- 2° $\lim_{t \rightarrow 0} \|f(x+t) - f(x)\| = 0$ uniformly in \mathfrak{F} .

Proof. Necessity. 1° is evident. If \mathfrak{F} is compact, then it is totally bounded. So for any $\epsilon > 0$, there are f_1, f_2, \dots, f_n in \mathfrak{F} such that for any $f \in \mathfrak{F}$ there is a k such as $\|f - f_k\| < \epsilon$. Since $\lim_{t \rightarrow 0} \|f(x+t) - f(x)\| = 0$, we have

$$\begin{aligned} \|f(x+t) - f(x)\| &= \|f(x+t) - f_k(x+t) + f_k(x+t) - f_k(x) + f_k(x) - f(x)\| \\ &\leq \|f(x+t) - f_k(x+t)\| + \|f_k(x+t) - f_k(x)\| + \|f_k(x) - f(x)\| \\ &\leq 3\epsilon. \end{aligned}$$

Thus we get 2°, and then the necessity of the condition.

Sufficiency. We suppose that $f(x+t) - f(x)$ is an abstract function of t whose range lies in E . Then the function is measurable in the Bochner sense⁵⁾. For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon)$ such that $\|f(x+t) - f(x)\| < \epsilon$ ($|t| < \delta$). Therefore $f(x+t) - f(x)$ is bounded and then it is integrable in the Bochner sense.

By 2°,

- 1) S. Izumi, Proc. **19** (1943), 99-101.
- 2) A. Kolmogoroff, Göttinger Nachrichten, (1931), 60-63.
- 3) A. Tulajkov, *ibid.*, (1933), 167-170.
- 4) M. Riesz, Acta Szeged, **6** (1932-34), 136-142.
- 5) S. Bochner, Fund. Math., **20** (1933), 262-276.