85. On the Strong Summability of Fourier Series.

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Let $f(x)$ be a real function of period 2π, integrable L over $(0, 2\pi)$, and let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

By $s_n(x)$ and $\sigma_n(x)$ we denote the n-partial sum and the n-th arithmetic mean of the above series, respectively.

Zygmund[1] has proved the following theorem.

If f is in L^p, where $p > 1$, then

$$\int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} (s_n - \sigma_n)^2 / n \right\}^{\frac{1}{2}p} dx \leqq A_p \int_0^{2\pi} |f|^p dx,$$

where A_p depends on p.

In §1, the author proves that the exponent 2 in the left hand side series may be replaced by arbitrary index $m \geqq 2$. In §2, we give a theorem on the strong summability of double Fourier series. The case of index $m=2$ has been given by Marcinkiewicz.[2] Finally in §3, the strong summability theorem of lacunary sequence of partial sums is proved. The case of index $m=2$ has been investigated by Zalcwasser[3] and Zygmund.[4]

I. We begin with some preliminary lemmas.[5]

Lemma 1. If $\{n_k\}$ denotes any sequence of positive integers satisfying the condition $n_{k+1}/n_k > a > 1$, then

$$\int_0^{2\pi} \left(\sum_{k=1}^{\infty} | s_{n_k} - \sigma_{n_k} |^2 \right)^{\frac{1}{2}p} dx \leqq B_p \int_0^{2\pi} |f|^p dx.$$

This is known.[6]

Lemma 2. Let $f_1, f_2, \dots$ be a sequence of functions of period 2π, integrable L, and let $s_{n,\nu}$ denotes the ν-th partial sum of the Fourier series of f_n. Then

$$\int_0^{2\pi} \left(\sum_{n=1}^{\infty} | s_{n,k_n} |^m \right)^{\frac{p}{m}} dx \leqq C_{m,p} \int_0^{2\pi} \left(\sum_{n=1}^{\infty} |f_n|^m \right)^{\frac{p}{m}} dx,$$

where $p > 1$ and $m > 1$.

This lemma is due to Boas and Bochner[7] when $k_n = \nu$. But the

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1) A. Zygmund, Fund. Math., **30** (1938), 170–196.
2) J. Marcinkiewicz, Annali di Pisa, **8** (1939), 149–160.
3) Z. Zalcwasser, Studia Math., **6** (1936), 82–88.
4) A. Zygmund, loc. cit.
5) $A_{m,p}, B_{m,p}, \dots$ denote constants depending only on m and p.
6) A. Zygmund, loc. cit.
7) R. P. Boas, Jr. and S. Bochner, Journ. London Math. Soc., **14** (1939), 62–73.