131. *Induced Measure Preserving Transformations.*

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§ **1.** The purpose of this paper is to investigate the properties of induced measure preserving transformations. We shall give only definitions and fundamental results, leaving the discussions of the details to another occasion.

§ **2.** A *measure space* $(\Omega, \mathfrak{B}, m)$ is a triple of a space $\Omega=\{\omega\}$, a Borel field $\mathfrak{B}=\{B\}$ of subsets B of Ω, and a countably additive measure $m(B)$ defined on $\mathfrak{B}$ satisfying $0 < m(\Omega) \leqq \infty$. In case $m(\Omega)=\infty$, it is assumed that there exists a sequence $\{B_n \mid n=1, 2, \dots\}$ of subsets B_n of Ω such that $B_n \in \mathfrak{B}$, $m(B_n) < \infty$, $n=1, 2, \dots$, and $\bigcup_{n=1}^{\infty} B_n = \Omega$. A subset B of Ω belonging to $\mathfrak{B}$ is called *$\mathfrak{B}$-measurable*, and $m(B)$ is its *m-measure*. A $\mathfrak{B}$-measurable subset N of Ω of m-measure zero is called a *null set* of $(\Omega, \mathfrak{B}, m)$, and the family of all null sets of $(\Omega, \mathfrak{B}, m)$ is denoted by $\mathfrak{N}(\Omega, \mathfrak{B}, m)$.

For any $\mathfrak{B}$-measurable subset Ω' of Ω with a positive m-measure, let us denote by $\mathfrak{B}_{\Omega'}$ the family of all $\mathfrak{B}$-measurable subsets B of Ω, and put $m_{\Omega'}(B)=m(B)$ on $\mathfrak{B}_{\Omega'}$. Then $(\Omega', \mathfrak{B}_{\Omega'}, m_{\Omega'})$ is a measure space which we call *the measure space induced on Ω' by* $(\Omega, \mathfrak{B}, m)$, or simply an *induced measure space.*

A one-to-one mapping $\omega'=\varphi(\omega)$ of a measure space $(\Omega, \mathfrak{B}, m)$ onto another measure space $(\Omega', \mathfrak{B}', m')$ is a *measure preserving transformation* (m. p. t.) *in a strong sense* if $B \in \mathfrak{B}$ implies $\varphi(B) \in \mathfrak{B}'$, $m'\big(\varphi(B)\big)= m(B)$ and if conversely $B' \in \mathfrak{B}'$ implies $\varphi^{-1}(B') \in \mathfrak{B}$, $m\big(\varphi^{-1}(B')\big)=m'(B')$. If there exists two null sets $N \in \mathfrak{N}(\Omega, \mathfrak{B}, m)$ and $N' \in \mathfrak{N}(\Omega', \mathfrak{B}', m')$, and if $\omega'=\varphi(\omega)$ is a m. p. t. in a strong sense of $(\Omega-N, \mathfrak{B}_{\Omega-N}, m_{\Omega-N})$ onto $(\Omega'-N', \mathfrak{B}'_{\Omega'-N'}, m'_{\Omega'-N'})$, then $\omega'=\varphi(\omega)$ is called a *measure preserving transformation* (m. p. t.) *in a weak sense* of $(\Omega, \mathfrak{B}, m)$ onto $(\Omega', \mathfrak{B}', m')$. $\mathfrak{D}(\varphi)=\Omega-N$ is the *domain* of φ and $\mathfrak{R}(\varphi)=\Omega'-N'$ is the *range* of φ. When we speak of a m. p. t. in a weak sense φ, the domain and the range of φ are usually not explicitly stated. Given two m. p. t. in a weak sense $\omega'=\varphi(\omega)$ and $\omega'=\psi(\omega)$ which map the same measure space $(\Omega', \mathfrak{B}', m')$ onto the same measure space $(\Omega', \mathfrak{B}', m')$, φ and ψ are called *almost equal* (notation: $\varphi \approx \psi$) if $\varphi(\omega)=\psi(\omega)$ almost everywhere on $(\Omega, \mathfrak{B}, m)$, or more precisely, if there exists a null set $N \in \mathfrak{N}(\Omega, \mathfrak{B}, m)$ such that $\Omega-N \subseteqq \mathfrak{D}(\varphi) \frown \mathfrak{D}(\psi)$ and $\varphi(\omega)=\psi(\omega)$ for all $\omega \in \Omega-N$. This notion of almost equality is clearly reflexive, symmetric and transitive. The class of all m. p. t. in a weak sense which are almost equal with φ is denoted by $[\varphi]$.

Let us now consider a case when two measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ coincide. Then we obtain the notion of a m. p. t. in a strong sense or in a weak sense $\omega'=\varphi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto itself. The family $\Phi(\Omega, \mathfrak{B}, m)$ of all m. p. t. in a strong sense of a