59. Note on Locally Compact Simple Rings.

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Taussky and Jacobson[1] proved that a locally compact separable simple ring is either connected or totally disconnected, and that in the first cast it is an algebra with finite bases over the field of all real numbers. I shall remark in this note that we can do without the assumption of separability.

Let R be an infinite locally compact (= bicompact) simple associative topological ring, where $x-y$ and xy are continuous.

a. *The component C of zero is a closed ideal in R.* The image aC of a connected C by the continuous mapping $x \rightarrow ax$ is also connected and contains zero, hence $aC \leq C$; similarly $Ca \leq C$. Therefore, the assertion follows from the well-known theorem that C is a closed additive subgroup of R.

b. *R is either connected or totally disconnected.* This is equivalent to $C=R$ or 0 and follows from the assumption that R is simple.

c. *If R is connected, R is a vector group A (of finite dimension).* By the theory of Pontrjagin-van Kampen, a locally compact connected additive group R is the direct sum of a vector group A and a compact group K: $R=A \oplus K$. *K is a closed ideal in R.* If this were not the case, there would, for example, exist $x_0 \in K$ and $c \in R$ such that $cx_0 \notin K$, i.e. $cx_0 = a+k$, $a \neq 0$, $a \in A$ and $k \in K$. Consider $na = c(nx_0) - nk$, $n=1, 2, \dots$. As a continous image of the pair of two compact sets is compact, $K' = cK - K$ is compact, and $c(nx_0) - nk \in K'$ $(n=1, 2, \dots)$. Hence $\{c(nx_0) - nk \mid n=1, 2, \dots\}$ would have a limit point, while otherhands $\{na \mid n=1, 2, \dots\}$ must not have limit point as $A \ni a \neq 0$. Therefore R = either A or K, as R is simple. But no compact topological ring is connected[2]. Thus $R=A$.

Now we have by Wedderburn's and Frobenius theorems

Theorem. *Every locally compact associative topological ring is either connected or totally disconnected. In the former case under the further assumption that it has the unit 1, it is a total matric algebra (of finite degree) over the field either of all real numbers or of complex numbers or of quaternions.*

Corollary. *All locally compact connected associative fields are completely classified into the three kinds of fields mentioned in Theorem (The continuity of division needs not be assumed.)*

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1) O. Taussky and N. Jacobson, Locally compact rings, Proc. Nat. Acad. Sci., **21** (1935).

2) See my preceding article "On quasi-evaluations of compact rings," Th. 1.