73. A Generalized Limit.

By Tsurane IWAMURA.

Mathematical Institute, Nagoya Imperial University.

(Comm. by T. TAKAGI, M.I.A., June 12, 1944.)

We are concerned with the definition and existence of a generalized limit, which possesses all the properties of the Banach limit in much generalized form except the invariance under translations of sequences. The invariance is closely related to the nature of the arithmetic mean.

Let Δ be a directed system, *i. e.* a partially ordered non-empty set, for any pair of whose elements α, β there exists a third element γ satisfying $\alpha \leqq \gamma$ and $\beta \leqq \gamma$. We shall say that a condition $C(\alpha)$ concerning a variable $\alpha \in \Delta$, is *finally* satisfied or that $C(\alpha)$ is satisfied *for large* α, when and only when there exists an $\alpha_0 \in \Delta$ such that $C(\alpha)$ is satisfied for all $\alpha \geqq \alpha_0$. If each of a finite number of conditions $C_1(\alpha), \dots, C_n(\alpha)$ is finally satisfied, their conjunction $C_1(\alpha) \& \dots C_n(\alpha)$ is also finally satisfied.

A *sequence* is a function $\varphi(\alpha)$ defined for large α in Δ. Let E be a Hausdorff space and let $\varphi(\alpha)$ be a sequence of its points: $\varphi(\delta) \in E$. A point $p \in E$ will be called the limit of the sequence, if and only if every neighbourhood of p finally contains $\varphi(\alpha)$. In such a case the set $\Phi(\alpha) = \overline{\{\varphi(\beta) \mid \beta \geqq \alpha\}}$ (bar indicates the closure in E) is finally bicompact, if E is locally bicompact. E will be supposed locally bicompact in what follows.

Let S be the totality of sequences $\varphi(\alpha)$ of points in E such that $\Phi(\alpha)$ are finally bicompact, and let $S' \leqq S$. A *generalized limit* of sequences $\varphi \in S'$ is a function $L(\varphi)$ defined for all $\varphi \in S'$, with values in E and satisfying the following conditions:

1) If $\varphi \in S'$ and $\Phi(\alpha) = \overline{\{\varphi(\beta) \mid \beta \geqq \alpha\}}$, then $L(\varphi) \in \Phi(\alpha)$.

2) If $\varphi, \psi \in S'$ and $\varphi(\alpha) = \psi(\alpha)$ for large α, then $L(\varphi) = L(\psi)$.

3) Let $\psi, \varphi_1, \dots, \varphi_n \in S'$ and let E^n be the cartesian product $E \times \dots \times E$ with the usual topology. If a function f defined in E^n, with values in E, is continuous at $\big(L(\varphi_1), \dots, L(\varphi_n)\big)$, and if $\psi \in S'$ where $\psi(\alpha) = f\big(\varphi_1(\alpha), \dots, \varphi_n(\alpha)\big)$ for large α, then $L(\psi) = f\big(L(\varphi_1), \dots, L(\varphi_n)\big)$.

It is easily seen (cf. 1) that $L(\varphi)$ coincides with the ordinary limit of φ, if the latter exists.

For the proof of existence of L, let us define $\varphi \equiv \psi$ when and only when $\varphi(\alpha) = \psi(\alpha)$ for large α, and let us first consider the case when for any distinct $\varphi, \psi \in S'$ $\varphi \equiv \psi$ does not hold. Let us further assume that for every $\varphi \in S'$ $\varphi(\alpha)$ is defined for all $\alpha \in \Delta$ and that $\overline{\varphi(\Delta)}$ is bicompact. The Hausdorff spaces $\overline{\varphi(\Delta)}$ being bicompact, their cartesian product $P = \prod_{\varphi \in S} \overline{\varphi(\Delta)}$ with weak topology is a bicompact Hausdorff space, which consists by definition, of all the functions $F(\varphi)$ defined for all