119. *Normed Rings and Spectral Theorems, VI.*

By Kôsaku YOSIDA.

Mathimatical Institute, Nagoya Imperial University.

(Comm. by T. TAKAGI, M.I.A., Oct. 12, 1944.)

1. *Introduction.* The arguments in 3 of the fifth note[1] was insufficient since the lemma 2 is valid for $z \in L_1(G)$ only. The purpose of the present note is i) to give a complete proof of (19)—the Plancherel's theorem—in 3 of the fifth note and ii) to show that the Bochner-Raikov's representation theorem[2] may be obtained easily from the Plancherel's theorem. In this way, the Fourier analysis may be subsumed under the operator theory in Hilbert space formulated in terms of the normed ring.

We will make use of, in this note, the results and the notations in the fifth note.

2. *Proof of (19).* The set $\left\{\varphi_z(\chi) = \lambda + \int_G x(g)\chi(g)dg\,;\ z = \lambda e + x,\ x \in L_1(G)\right\}$ is dense in the space $C(X \cup \chi_\infty)$ of continuous functions $T(\chi)$ on the character group X of G compactified by adjoining the formal character χ_∞. This results from the Gelfand-Silov's abstraction[3] of Weierstrass' polynomial approximation theorem. We have

$$\| T_z \| = \lim_{n\to\infty} \sqrt[n]{\| T_z^n \|} \leqq \lim_{n\to\infty} \sqrt[n]{\| z^n \|_1} = \sup_x |\varphi_z(\chi)|$$

by $\| T_z \| \leqq \| z \|_1$. Hence,

$$(*)\quad \begin{cases} \text{for any continuous function } T(\chi) \text{ on } X \cup \chi_\infty, \text{ there exists one} \\ \text{and only one operator } T \text{ such that } \lim\limits_{n\to\infty} \sup\limits_{\chi} |\varphi_{z_n}(\chi) - T(\chi)| = 0 \\ \text{implies } \lim\limits_{n\to\infty} \| T_{z_n} - T \| = 0. \end{cases}$$

Let $x \in L_1(G) \frown C(G)$ and put

$$J\big(\varphi_x(\chi)\big) = x(0)\,.$$

J is additive, homogeneous and positive on $\{\varphi_x(\chi)\}$:

$$\varphi_x(\chi) \geqq 0 \qquad \text{implies} \qquad J\big(\varphi_x(\chi)\big) \geqq 0\,.$$

The proof was given by the lemma 3. The Plancherel's theorem may be proved if we show that

$$(**)\qquad J\big(\varphi_x(\chi)\big) = \int_X \varphi_x(\chi)d\chi\,,\quad d\chi = \text{the Haar's measure on } X.$$

This formula together with the definition

---

1) Proc. **20** (1944), 269.

2) C. R. URSS: **28**, 4 (1940).

3) Rec. Math., **9** (51), (1941).