133. *On the Representation of Functions by Fourier Integrals.*

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1. *Introduction and the theorems.* The purpose of the present note is to give the following representation theorems of complex-valued bounded continuous functions $f(t)$ on $(-\infty, \infty)$. The theorems may be applied in Fourier analysis as well as in probability theory. Since the proofs are carried through by virtue of the Plancherel's duality theorem, our results may be extended to the case of separable, locally compact abelian groups instead of the infinite line $(-\infty, \infty)$[1].

Theorem 1. $f(t)$ is positive definite[2] if and only if

$$\varphi_n(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left(\frac{\sin\frac{t}{n}}{\frac{t}{n}}\right)^2 e^{-it\lambda}dt\geqq 0 \tag{1}$$

$$(n=1, 2, \dots),$$

and if (1) is satisfied, we have the representation[3]:

(2) $\begin{cases} f(t)=\int_{-\infty}^{\infty}e^{it\lambda}dv(\lambda) \quad \text{with a monotone increasing, right-con-} \\ \text{tinuous bounded function } v(\lambda). \end{cases}$

Theorem 2.[4] $f(t)$ is positive definite if and only if $f(t)$ is expressible as

(3) $\begin{cases} f(t)=\lim\limits_{n\to\infty}\int_{-\infty}^{\infty}g_n(t+s)\overline{g_n(s)}ds \quad \text{uniformly in every finite inter-} \\ \text{val of } t, \text{ where} \end{cases}$

$$\sup_{n\geq 1}\int_{-\infty}^{\infty}|g_n(t)|^2dt\leqq f(0). \tag{4}$$

Theorem 3. $f(t)$ is representable in the form:

(5) $\begin{cases} f(t)=\int_{-\infty}^{\infty}e^{it\lambda}dv(\lambda) \quad \text{with a complex-valued right-continuous} \\ \text{function } v(\lambda) \text{ of bounded variation,} \end{cases}$

if and only if

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1) Cf. Proc. **20** (1944), 560–563.

2) $\overline{f(-t)}=f(t)$ and $\sum^{n} f(t_j-t_k)\xi_j\bar{\xi}_k\geqq 0$ for any integer n and for arbitrary complex numbers ξ.

3) S. Bochner: Vorlesungen über Fouriersche Integrale, Leipzig (1932), 76.

4) A. Khintchine: Bullt. de l'universitê d'état à Moscou, Sect. A, vol. **1**, fasc. 5, 1–3.