143. *Two-dimensional Brownian Motion and Harmonic Functions.*

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1. The purpose of this paper is to investigate the properties of two-dimensional Brownian motions[1] and to apply the results thus obtained to the theory of harmonic functions in the Gaussian plane. Our starting point is the following theorem: *Let D be a domain in the Gaussian plane R^2, and let E be a closed set on the boundary $Bd(D)$ of D. Then, under certain assumptions on D and E, the probability $P(\zeta, E, D)$, that the Brownian motion starting from a point $\zeta \in D$ will enter into E without entering into the other part $Bd(D)-E$ of the boundary of D before it, is equal to the harmonic measure in the sense of R. Nevanlinna of E with respect to the domain D and the point ζ.*

It is expected that, by means of this method, many of the known results in the theory of harmonic or analytic functions will be interpreted from the standpoint of the theory of probability. We shall here give only the fundamental results and a few of its applications, leaving the detailed discussions of further applications to another occasion.

Most of the results obtained in this paper are also valid for the case of higher dimensional Brownian motions. But there are also many theorems in which the dimension number plays an essential rôle[3]. For example, Theorems 6, 7 and 8 of this paper are no longer true in R^3. The situation will become clearer if we observe the following theorem: *Consider the n-dimensional Brownian motion in $R^n (n \geqq 2)$, and let $\bar{K}^n$ be the closed unit sphere in R^n. Then, for any $\zeta \in R^n - \bar{K}^n$, the probability $P(\zeta, \bar{K}^n, R^n - \bar{K}^n)$ that the Brownian motion starting from ζ will enter into $\bar{K}^n$ for some $t > 0$ is equal to $|\zeta|^{2-n}$ if $n \geqq 3$*[4]*, while this probability is $\equiv 1$ on $R^n - \bar{K}^n$ if $n=2$*[5]. This result is closely related with the fact that there is no bounded harmonic function, other than the constant 1, which is defined on $R^n - \bar{K}^n$ and tends to 1 as $|\zeta| \to 1$, while, for any $n \geqq 3$, $u(\zeta) = |\zeta|^{2-n}$ is a non-trivial example of a bounded harmonic function with the said property.

2. Let $\{z(t, \omega) = \{x(t, \omega), y(t, \omega)\} \mid -\infty < t < \infty, \omega \in \Omega\}$ be a two-dimensional Brownian motion defined on the $z=\{x, y\}$-plane R^2, i.e. an independent system of two one-dimensional Brownian motions $\{x(t, \omega) \mid$

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1) Brownian motions were discussed by N. Wiener and P. Lévy. Cf. N. Wiener, Generalized harmonic analysis, Acta Math., **54** (1930); N. Wiener, Homogeneous chaos. Amer. Journ. of Math., **60** (1939); R. E. A. C. Paley and N. Wiener, Fourier transforms in the complex domain, New York, 1933; P. Lévy, L'addition des variables aléatoires, Paris, 1937; P. Lévy, Sur certains processus stochastiques homogènes, Compositio Math., **7** (1939); P. Lévy, Le mouvement brownien plan, Amer. Journ. of Math., **61** (1940).

2) Cf. R. Nevanlinna, Eindeutige analytische Funktionen, Berlin, 1937.

3) Cf. S. Kakutani, On Brownian motions in n-space, Proc. **20** (1944).

4) $|\zeta|$ denotes the Euclidean distance of ζ from the origin of R^n.

5) The case $n=2$ is contained in Theorem 4 of this paper.