The use of norm attainment

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1 Introduction

The purpose of this short note is to gather a number of natural examples, belonging to various domains of analysis, where a result from fundamental functional analysis provides concrete inequalities in a unified manner. A familiar technique in this respect is to apply Baire category theorem via the uniform boundedness principle. This note displays arguments of a different kind, where the leading role is played by James' fundamental characterization of weak compactness [J], and Simons' inequality [S] which proves it in the separable case (see [Pf1], [Pf2] for recent and deep progress in the non-separable frame).

Let X be a Banach space, and let S be a norm-closed subspace of the dual space X^*. We denote B_S its closed unit ball. The space S is called separating if : $x^*(x) = 0$ for all $x^* \in S$ implies that $x = 0$. It is called norming if the functional $N(x) = \sup\{|x^*(x)|;\ x^* \in B_S\}$ is an equivalent norm on X and it is called 1-norming if N is equal to the original norm on X.

Straightforward applications of the Hahn-Banach theorem show that S is separating if and only if it is weak* dense in X^*, and that it is norming if and only if the weak* closure of B_S contains λB_{X^*} for some $\lambda > 0$, and finally that it is 1-norming if and only if $\lambda = 1$, in other words if and only if this weak* closure is equal to B_{X^*}. Easy examples show that these three notions are distinct : for instance, any hyperplane $H = Ker(x^{**})$ with $x^{**} \in X^{**} \backslash X$ is norming, but it is 1-norming if and only if $\|x + x^{**}\| \geqslant \|x\|$ for all $x \in X$. Also, take $X = c_0(\mathbf{N})$ equipped with its natural norm, and write $\mathbf{N}$ as a disjoint union of infinite sets $\bigcup_{j \geqslant 0} I_j$ with $0 \notin I_0$. Let S be the subspace of $l_1(\mathbf{N})$ consisting of all $y = (y_n)$ such that for all $k \in I_0$

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