A Phragmén - Lindelöf principle for slice regular functions

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1 Introduction

The celebrated 100-year old Phragmén-Lindelöf theorem, [19, 20], is a far reaching extension of the maximum modulus theorem for holomorphic functions. In its simplest form, it can be stated as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain whose boundary contains the point at infinity. If f is a bounded holomorphic function on Ω and $\limsup_{z\to z_0} |f(z)| \leq M$ at each finite boundary point z_0, then $|f(z)| \leq M$ for all $z \in \Omega$.*

The term Phragmén-Lindelöf also applies to a number of variations of this result, which guarantee a bound for holomorphic functions, when conditions are known on their growth. The two most famous variations deal with functions which are holomorphic in an angle or in a strip, and they can be stated as follows (see, for instance, [4, 17] as well as [1, 12]).

Theorem 1.2. *Let f be a holomorphic function on an angle Ω of opening $\frac{\pi}{\alpha}$. Suppose f is continuous up to the boundary and such that, for some $\rho < \alpha$, $|f(z)| \leq \exp(|z|^\rho)$ asymptotically. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial\Omega$ then $|f| \leq M$ in Ω.*

Theorem 1.3. *Let f be a holomorphic function on a strip Ω of width 2γ, continuous up to the boundary. Suppose that $|f(z)| \leq N \exp(e^{k|z|})$ in Ω for some positive constants N and $k < \frac{\pi}{2\gamma}$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial\Omega$ then $|f| \leq M$ in Ω.*

*Partially supported by GNSAGA of the INdAM and by PRIN "Proprietà geometriche delle varietà reali e complesse" of the MIUR.

†Partially supported by PRIN "Geometria Differenziale e Analisi Globale" of the MIUR.

Received by the editors November 2010.

Communicated by F. Brackx.

2000 *Mathematics Subject Classification* : 30G35, 30C80.

Key words and phrases : Phragmén-Lindelöf, slice regular functions of a quaternionic variable.