DIRECT AND SUBDIRECT FACTORIZATIONS OF LATTICES

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(Received Dec. 13, 1950)

Let a lattice L be a product[1] $L_1 \dots L_n$ of n lattices L_i $(i=1, \dots, n)$. If L has the null element 0 and the unit element 1, then L_i has the null element 0_i and the unit element 1_i. The element z_i which is expressed in $L_1 \dots L_n$ as $[0_1, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n]$ is an element of the center of L[2]. The center of L is a Boolean algebra, and using this center we can easily solve the factorization problem of lattices[3]. But for the lattices L without 0 or 1, the centers of L do not exist. Hence for the factorization problem of such lattices, we must seek Boolean algebras. From this point of view, I investigated the direct factorizations and the subdirect factorizations of lattices without the assumption that 0 and 1 exist.

§ I. Direct Factorizations of Lattices.

By a *direct factorization* of a lattice L we mean the system of lattices L_i $(i=1, \dots, u)$, when L is isomorphic to the product $\Pi(L_i; i=1, \dots, n) = L_1 \dots L_n$. Let $\Theta(L)$ denote the set of all congruence relations on L. Funayama and Nakayama proved that $\Theta(L)$ is an upper continuous, distributive lattice by defining $\theta \leqq \phi$ if and only if $x \equiv y\,(\theta)$ implies $x \equiv y\,(\phi)$[4]. Two congruence relations θ and ϕ are called *permutable* if $a \equiv x\,(\theta)$ and $x \equiv b\,(\phi)$ for some x imply $a \equiv y\,(\phi)$ and $y \equiv b\,(\theta)$ for some y. The set of all congruence relations which are permutable with θ for all $\theta \in \Theta(L)$ is denoted by $\Gamma(L)$. And the center of $\Theta(L)$ is denoted by $\Theta_z(L)$. Since $\Theta(L)$ is distributive, $\theta \in \Theta_z(L)$ if and only if θ has its complement θ'. If $L \cong L_1 L_1$, the mapping $[x_1, x_2] \to x_1$ is a homomorphism of L onto L_1 and hence generates a congruence relation θ_1, which we call a *decomposition congruence relation*. If we denote by $\Theta_0(L)$ the set of all decomposition

1) Cardinal product in Birkhoff's [1, p. 25] sense. The numbers in square brackets refer to the list at the end of this paper.

2) Center in Birkhoff's [1, p. 27] sense.

3) Cf. Birkhoff [1] 26.

4) Cf. Birkhoff [1] 24. A complete lattice L is called *upper continuous* when $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$. When L is distributive, this is equivalent to $V(a; a \in S) \frown b = V(a \frown x; a \in S)$ for all $S \leqq L$. We use also 0 and 1 for the zero element and the unit element of $\Theta(L)$ respectively.