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Ricci tensors on unit tangent sphere bundles over 4-dimensional Riemannian manifolds

Jong Taek Cho and Sun Hyang Chun

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Abstract. For a 4-dimensional Riemannian manifold (M, g) let T_1M be its unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Then we prove that the Ricci operator S and the structure operator ϕ commute i.e., $S\phi = \phi S$ (anti-commute i.e., $S\phi + \phi S = 2k\phi$, respectively) if and only if (M, g) is of constant sectional curvature 1 or 2 ((M, g) is of constant sectional curvature, respectively).

1. Introduction

It is intriguing to study the interplay between Riemannian manifolds and their unit tangent sphere bundles. In particular, we are interested in the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ of a unit tangent sphere bundle T_1M over a given Riemannian manifold (M, g). As a classical result, Tashiro ([13]) proved that $(T_1M, \eta, \bar{g})$ is a K-contact manifold (i.e., the Reeb vector field ξ is a Killing vector field) if and only if (M, g) has constant sectional curvature 1.

Boeckx and Vanhecke ([4]) proved that T_1M is Einstein, that is $\bar{\rho} = \alpha\bar{g}$ if and only if (M, g) is 2-dimensional and is locally isometric to the Euclidean plane or the unit sphere, where $\bar{\rho}$ denotes the Ricci curvature tensor of T_1M and α is a function of T_1M. In [6], for a 4-dimensional Riemannian manifold M it was proved that T_1M is η-Einstein, that is $\bar{\rho} = \alpha\bar{g} + \beta\eta \otimes \eta$ if and only if M is of constant sectional curvature 1 or 2, where α, β are functions of T_1M. Later, Park and Sekigawa ([9]) generalized the result for higher dimensional cases. In fact, they proved that T_1M is η-Einstein if and only if (M^n, g) is of constant sectional curvature 1 or $n-2$, where $\dim M = n$. After all, we are aware that (η-)Einstein condition is too strong to impose on T_1M. This

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The second author is the corresponding author.

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