J. Sci. Hiroshima Univ. Ser. A-I
26 (1962) 1–2

Note on a Characterization of Solvable Lie Algebras

Morikuni Goto

(Received January 24, 1962)

By a Lie algebra we mean a finite-dimensional Lie algebra over a field of characteristic 0. Our main purpose in this note is to establish the following Theorem.

Theorem.[1] *Let* $\mathfrak{g}$ *be a Lie algebra. If there exist nilpotent subalgebras* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *with* $\mathfrak{n}_1+\mathfrak{n}_2=\mathfrak{g}$, *then* $\mathfrak{g}$ *is solvable, and vice versa.*

The proof of the Theorem depends on the following Lemma.

Lemma. *Let* $\mathfrak{s}$ *be a semi-simple Lie algebra, and let* $\mathfrak{n}$ *be a nilpotent subalgebra of* $\mathfrak{s}$. *Let* r *be the dimension of* $\mathfrak{s}$, *and let* l *be the rank of* $\mathfrak{s}$. *Then we have the following inequality*:

$$2 \dim \mathfrak{n} \leq r - l.$$

About the concept and results in the theory of Lie algebras, used in this note, the reader may refer: e.g. C. Chevalley, Théorie des groupes de Lie, t. III, Hermann, Paris, 1955.

(A) *Proof of the Lemma*

Let $\mathfrak{s}$ be a semi-simple Lie algebra over an algebraically closed field. Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{s}$. We use the notation $\mathrm{ad}(X)Y=[X, Y]$ for $X, Y \in \mathfrak{s}$. An element X is said to be *semi-simple* if $\mathrm{ad}(X)$ is a semi-simple linear transformation on $\mathfrak{s}$. Let A be a semi-simple element in $\mathfrak{n}$. Since $\mathrm{ad}(X)$, restricted on $\mathfrak{n}$, is nilpotent for any X in $\mathfrak{n}$, we have $\mathrm{ad}(A)=0$ on $\mathfrak{n}$, i.e. A is in the center of $\mathfrak{n}$. We denote by $\mathfrak{a}$ the set of all semi-simple elements in $\mathfrak{n}$. Then $\mathfrak{a}$ is a central ideal of $\mathfrak{n}$.

Let $\mathfrak{m}$ be a maximal solvable subalgebra of $\mathfrak{s}$ containing $\mathfrak{n}$. We can find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ such that

$$\mathfrak{m} \supset \mathfrak{h} \supset \mathfrak{a}.$$

Since any element of $\mathfrak{h}$ is semi-simple, we have $\mathfrak{n} \cap \mathfrak{h}=\mathfrak{a}$. Next, because $\mathfrak{a}$ is a commutative subalgebra composed of semi-simple elements, we obtain a decomposition of $\mathfrak{m}$ into submoduli:

$$\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2,\ \mathfrak{m}_1 \cap \mathfrak{m}_2=\{0\},$$

$$[\mathfrak{a}, \mathfrak{m}_1]=\{0\}, \text{ and } [\mathfrak{a}, \mathfrak{m}_2]=\mathfrak{m}_2.$$

---

1) It seems that the question whether the theorem is true or not had been raised originally by Otto H. Kegel and the author was inquired about this by S. Tôgô.