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Evaluation of Hausdorff Measures of Generalized Cantor Sets

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§1. Introduction

The problem how a Hausdorff measure of a product set $A \times B$ is related to Hausdorff measures of A and B is not completely solved. This problem was first investigated by F. Hausdorff himself [3] and later by A. S. Besicovitch and P. A. P. Moran [1], J. M. Marstrand [4] and others. Their works and investigations of similar problem for capacity (e.g. [6], [7]) show that evaluation of Hausdorff measures of generalized Cantor sets supplies many clues to this problem.

In this paper we first evaluate the α-Hausdorff measure of generalized Cantor sets in the Euclidean space R^n. As a concequence we see the existence of a compact set in R^n which has infinite α-Hausdorff measure but zero α-capacity $(0<\alpha<n)$. Next we estimate Hausdorff measures of product sets of one-dimensional generalized Cantor sets and then give examples which show that in case the α-Hausdorff measure of E_1 is infinite and the β-Hausdorff measure of E_2 is zero, the $(\alpha+\beta)$-Hausdorff measure of $E_1 \times E_2$ may either be zero, positive finite or infinite. Also these examples answer M. Ohtsuka's question in [7] (p. 114) in the negative.

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§2. Definitions and Notation

Let $R^n (n \geqq 1)$ be the n-dimensional Euclidean space with points $x=(x_1, x_2, \cdots, x_n)$. By an n-dimensional open cube (closed cube resp.) in R^n, we mean the set of points $x=(x_1, x_2, \cdots, x_n)$ satisfying the inequalities:

$$a_i < x_i < a_i + d \ (a_i \leqq x_i \leqq a_i + d \text{ resp.}) \qquad \text{for} \quad i=1, 2, \cdots, n,$$

where a_i $(i=1, 2, \cdots, n)$ are any numbers and $d>0$. We call d the length of the side, or simply the side, of the open (or closed) cube.

Let $\mathfrak{A}$ be the family of non empty open sets in R^n which is determined by the following properties:

(i) any n-dimensional open cube belongs to $\mathfrak{A}$,

(ii) if ω_1 and ω_2 belong to $\mathfrak{A}$, then so does $\omega_1 \cup \omega_2$,

(iii) if ω is an element of $\mathfrak{A}$, then there exists a finite number of n-