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Schwarz Reflexion Principle in 3-Space

Makoto Ohtsuka

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Introduction

The Schwarz reflexion principle is well-known in the theory of harmonic functions in a plane. In the three dimensional euclidean space ($=$3-space), however, it seems that some problems remain to be discussed.[1] In this paper, we shall show that any harmonic function h, defined in a domain D within an open ball V and having vanishing normal derivative on a part E of $\partial D \cap \partial V$, can always be continued across E but in general only *radially*.

J. W. Green [2] treated the case where D coincides with V. He showed that h is continued harmonically through E to the entire outside of V if and only if $\int_0^R h(r, \theta, \varphi)dr$ is constant as a function of (θ, φ) on the set $\{(\theta, \varphi); (R, \theta, \varphi) \in E\}$, and that there is a case where h cannot be continued harmonically to the entire outside of V.

§**1.** First we explain notation. Throughout this paper, V means the open ball with center at the origin 0 and radius R in the 3-space, $S=\partial V$ its boundary, D a subdomain of V, ∂D its boundary, E a two dimensional open set on $\partial D \cap S$ which contains no point of accumulation of $\partial D - E$, h a harmonic function in D, and, for a point $P \in D$, P' the symmetric point of P with respect to S. This point is called also the point of reflexion or the mirror image of P.

The case when h vanishes on E is known and stated as

Proposition. *If h is continuous on $D \cup E$ and vanishes on E, then h is extended through E to a harmonic function in the domain D' which is the reflexion of D with respect to S.*

Proof. Choose any $Q \in S$ and let Σ be the spherical surface with center Q and radius R_0. Invert the space with respect to Σ and denote by P^* the image of P by the inversion. The image of S is a plane, and P^* and P'^* are symmetric with respect to the plane. Define a function $h^*(P^*)$ by $\overline{OQ} \cdot h(P)/R_0$

1) O. D. Kellogg suggested to "derive results similar to (the result in the case where $h=0$ on E), where $\cdots$ it is assumed that the normal derivative of U vanishes on that portion" in Exercise 4 at p. 262 of [3]. It is stated at p. 244 in Lichtenstein [4] that "$\cdots$ (plane case) $\cdots$. Analoge Sätze gelten im Raume." However, this turns out not to be the case.