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On Overrings of a Domain

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1. Introduction

Throughout this paper D will denote an integral domain with $1\neq 0$ and quotient field K, and by an overring of D will be meant a ring J such that $D\subset J\subset K$. An ideal A of D is called a valuation ideal provided there exists a valuation overring D_V of D such that $AD_V\cap D=A$ ([22; 340], [10]). If Π is a general ring property, then we shall refer to an ideal A of D as a *Π-ideal* provided there exists an overring J of D such that J is a *Π-domain* (i.e. J has the property Π) and $A=AJ\cap D$. It is shown in [10] that if every principal ideal of D is a valuation ideal, then D is a valuation ring. Furthermore, if every proper ideal of D is a Dedekind ideal, then D is a Dedekind domain [2]; and if every proper ideal of D is a Prüfer ideal, then D is a Prüfer domain [7], [10; 238]. In this paper we are mainly concerned with the following question. When does the statement

(a) "there exists a collection $\mathcal{S}$ of Π-ideals of D" imply the statement

(b) "D is a Π-domain" (i.e. D has property Π)? Our main result in this direction is that (a) implies (b) when "Π-domain" = "Krull domain" and $\mathcal{S}$ is the collection of proper principal ideals of D, i.e. if every proper principal ideal is a Krull ideal, then D is a Krull domain. The same result holds in case "Krull domain is replaced by either "integrally closed domain" or "completely integrally closed domain". In addition we show that (a) implies (b) when $\mathcal{S}$ is the collection of proper finitely generated ideals of D and Π is any of the following ring properties: Prüfer, 1-dim. Prüfer, almost Dedekind, or Dedekind.

We remark that (a) does not always imply (b), even in the case that $\mathcal{S}$ is the set of all ideals of D (e.g. if Π is one of *P.I.D.*, Bezout, or *QR*-property-see Section 5).

In general we use the notation and terminology of [21] and [22]. In particular, $\subset$ denotes containment, while $<$ denotes proper containment; and A is a proper ideal of D provided $(0)<A<D$. The theorems considered in this paper are trivial in case D is a field, so we assume throughout that D has at least one proper ideal.

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