J. Sci. HIROSHIMA UNIV. SER. A-I
33 (1969), 209-212

A Theorem on Characteristically Nilpotent Algebras

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(Received September 20, 1969)

1. The notion of characteristic nilpotency of Lie algebras has been introduced by Dixmier and Lister [3] and studied by Leger and Tôgô [4]. Let L be a Lie algebra over a field Φ and $\mathfrak{D}(L)$ be the set of all derivations of L. Put $L^{[1]}=L\mathfrak{D}(L)=\{\sum a_iD_i \mid a_i \in L,\ D_i \in \mathfrak{D}(L)\}$ and define inductively $L^{[n]}=L^{[n-1]}\mathfrak{D}(L)$ for $n \geq 2$. L is called characteristically nilpotent provided there exists an integer k such that $L^{[k]}=(0)$. Then every characteristically nilpotent Lie algebra is nilpotent. However, few similarities are known between the properties of nilpotent and characteristically nilpotent algebras, e.g., a subalgebra and a quotient algebra of a characteristically nilpotent Lie algebra are not necessarily characteristically nilpotent, contrary to the case of nilpotent algebras.

Recently, in [2], C.-Y. Chao has shown a characterization of nilpotent Lie algebras: Let L be a Lie algebra over a field Φ and N be a nilpotent ideal of L. Then L is nilpotent if and only if L/N^2 is nilpotent.

The purpose of this note is to show a similar characterization of characteristically nilpotent Lie algebras, as a matter of fact, more generally of characteristically nilpotent nonassociative algebras.

By a nonassociative algebra we mean an algebra which is not necessarily associative, that is, a distributive algebra [6]. The definition of characteristic nilpotency of a nonassociative algebra A is obtained by replacing A instead of L in that of a Lie algebra stated above and this is due to the first version of the paper [5] of T.S. Ravisankar. However, it has not yet been known that there actually exists a characteristically nilpotent nonassociative algebra which is not a Lie algebra. We shall show the existence of such an algebra in Section 3. All the algebras considered in this note are assumed to be finite dimensional over their base fields.

2. For a nonassociative algebra N, all the products of n elements in N, irrespective of how they are associated, span a subspace of N, which is a subalgebra of N and denoted by N^n. N is called nilpotent if $N^k=(0)$ for some k [6].

LEMMA. *Let A be a nonassociative algebra over a field Φ and N be a characteristic subalgebra of A. If $N\mathfrak{D}(A)^m \subset N^n$, then for every integer $r \geq 1$*

$$N^r\mathfrak{D}(A)^{rm-r+1} \subset N^{n+r-1}.$$

PROOF. We prove the statement by induction on r. It holds for $r=$