HIROSHIMA MATH. J.
11 (1981), 415–423

Central limit theorem for a simple diffusion model of interacting particles

Hiroshi TANAKA and Masuyuki HITSUDA

(Received January 19, 1981)

§ 1. Introduction

Given a smooth function $b(x, y)$, $x, y \in \boldsymbol{R}$, we set

$$b[x, u] = \int_{\boldsymbol{R}} b(x, y)u(y)dy \qquad \left(\text{or} = \int_{\boldsymbol{R}} b(x, y)u(dy)\right)$$

for a function $u(y)$ (or a measure $u(dy)$), and consider

$$\text{(1.1a)} \qquad \frac{\partial u}{\partial t} = \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(b[x, u]u)\,,$$

$$\text{(1.1b)} \qquad u(0, x) = u(x)\,,$$

where $u(x)$ is a probability density function. In connection with Kac's work [4] on the propagation of chaos for Boltzmann's equation, McKean [7] described the diffusion process $\{X(t)\}$ associated with (1.1) as the limit process, as $n \to \infty$, of any single component process of the diffusion $X^{(n)}(t) = (X_1^{(n)}(t), \dots, X_n^{(n)}(t))$ with generator

$$\text{(1.2)} \qquad K^{(n)}\varphi = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 \varphi}{\partial x_i^2} + \sum_{i=1}^n \left(\frac{1}{n-1}\sum_{k \neq i} b(x_i, x_k)\right)\frac{\partial \varphi}{\partial x_i}$$

and with initial density $u(x_1)u(x_2)\cdots u(x_n)$; in fact, it was shown that for each fixed m the process $\{(X_1^{(n)}(t), \dots, X_m^{(n)}(t))\}$ converges in law to $\{(X_1(t), \dots, X_m(t))\}$ where $\{X_k(t)\}$, $k = 1, 2, \dots$, are independent copies of $\{X(t)\}$. Thus we have the following law of large numbers:

$$\text{(1.3)} \qquad U^{(n)}(t) = n^{-1}\sum_{k=1}^n \delta_{X_k^{(n)}(t)} \longrightarrow u(t)\,;$$

here δ_x denotes the δ-distribution at x and $u(t) = u(t, x)dx$ where $u(t, x)$ is the solution of (1.1). The next stage is the central limit theorem which investigates the limit of

$$\text{(1.4)} \qquad S^{(n)}(t) = n^{1/2}(U^{(n)}(t) - u(t))\,,$$

as $n \to \infty$. This kind of problem was considered by Kac [5] and McKean [8] for Boltzmann's equation, and by Martin-Löf [6] and Itô [3] for non-interacting