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Classes of generalized soluble Lie algebras

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Introduction

A class $\mathfrak{X}$ of Lie algebras is said to be a class of generalized soluble Lie algebras if every soluble Lie algebra is an $\mathfrak{X}$-algebra and every finite-dimensional $\mathfrak{X}$-algebra is soluble. As relatively large classes of generalized soluble Lie algebras we know the classes $\hat{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\hat{\mathrm{E}}\mathfrak{A}$, which are the Lie-theoretic analogues of the class of SI-groups and the class of SN-groups respectively. In group theory Mal'cev [6] has proved that the class of SI-groups, the class of SN-groups and the class of Z-groups are L-closed. The first purpose of this paper is to prove the Lie-theoretic analogue of this result.

Generalizing the class $\mathfrak{R}$ of residually central Lie algebras, Amayo [2] has introduced a relatively large class, denoted by $\mathfrak{R}^{(1)}$ in this paper, of generalized soluble Lie algebras. In the recent paper [5] we have introduced the class $\mathfrak{R}_{(\infty)}$ of residually (ω)-central Lie algebras. The second purpose of this paper is to introduce and investigate various classes of Lie algebras generalizing the class $\mathfrak{R}$. Most of them are classes of generalized soluble Lie algebras.

In Section 2, following [8, §8.2] it can be more generally proved that the classes $\hat{\mathrm{E}}\mathfrak{A}$, $\hat{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$ are L-closed, where $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$ is the class of Lie algebras having central series (Theorems 2.2 and 2.6). We shall also show that every finite-dimensional subalgebra of an $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$-algebra (resp. a hypocentral Lie algebra) is serial (resp. descendant) (Theorem 2.9).

In Section 3 we shall develop some results analogous to those of [5, §2] by using the class $\mathfrak{R}_{(*)}$, naturally including the class $\mathfrak{R}_{(\infty)}$, of generalized soluble Lie algebras. Especially, we shall show that $\mathfrak{R}_{(*)} \cap \mathfrak{M}^{(*)} = \grave{\mathrm{E}}\mathfrak{A}$ (Theorem 3.5), where $\mathfrak{M}^{(*)}$ is a class of Lie algebras generalizing quasi-artinian Lie algebras.

Section 4 is devoted to investigating the classes $\mathfrak{R}^*$, $\mathfrak{R}^{(*)}$, $\mathfrak{R}^{(1)}_{(*)}$ and $\mathfrak{R}^{(*)}_{(*)}$, naturally including the class $\mathfrak{R}^{(1)}$, of generalized soluble Lie algebras. We shall show that $\mathfrak{R}^{(1)} = \mathfrak{R}^* = \mathfrak{R}^{(*)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(1)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^* = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(*)}$ and $\mathfrak{R}^{(1)}_{(*)} = \mathfrak{R}^{(*)}_{(*)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(1)}_{(*)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(*)}_{(*)}$ (Theorem 4.3). We shall also show that $\mathfrak{R}^{(*)}_{(*)} \cap \text{Min-}\lhd = \acute{\mathrm{E}}(\lhd)\mathfrak{A} \cap \text{Min-}\lhd$ (Theorem 4.6).

In Section 5 we shall investigate the classes $\mathfrak{R}_{(1)}$ and $\mathfrak{R}_*$ which are between the classes $\mathfrak{R}$ and $\mathfrak{R}_{(*)}$. In particular, we shall present a sufficient condition for a Lie algebra to be contained in the class $\mathfrak{R}^{(1)}$ and consequently show that $\mathfrak{R}_{(1)}$ is a subclass of the class $\mathfrak{R}^{(1)}$ (Theorem 5.2).