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On exterior A_n-spaces and modified projective spaces

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1. Introduction

A space X with a continuous multiplication $\mu: X \times X \to X$ with a unit is called an H-space. A typical example of H-space is a loop space. It is known that not all H-spaces have the homotopy type of loop spaces. The 7-dimensional sphere S^7 is one of such counter examples.

Sugawara [12] gave a criterion for an H-space to have the homotopy type of a loop space. His criterion is a kind of higher homotopy associativity of infinite order. Almost the same time Stasheff [9] reached the same idea, and he defined the A_n-space which is the H-space with higher homotopy associative multiplication of n-th order. In his sense A_2-spaces are H-spaces, A_3-spaces are homotopy associative H-spaces, and A_∞-spaces are spaces with the homotopy type of loop spaces.

In his paper, Stasheff defined the projective n-space $P_n(X)$ associated to a given A_n-space X, which is considered as a generalization of the n-th stage of the construction of the classifying space of a topological group or an associative H-space. In fact, $P_n(X)$ is defined inductively by $P_n(X) = P_{n-1}(X) \cup C(X^{*n})$ with $P_0(X) = *$, where X^{*n} is the n-fold join of X. Then Stasheff proved that if $X = \Omega Y$, then $P_\infty(X)$ has the homotopy type of Y, where $P_\infty(X) = \bigcup_{i=1}^{\infty} P_i(X)$. The name 'projective' comes from the fact that if X is the unit sphere in the real, the complex or the quaternionic numbers, then $P_n(X)$ is the usual real, complex or quaternionic projective n-space.

The projective n-space has been very useful for the study of the cohomology of A_n-spaces. In fact, we have the following fact.

THEOREM (Iwase [4]). *Let X be a simply connected A_n-space so that*

$$H^*(X; \mathbf{Z}/p) \cong \Lambda(x_1, \dots, x_k), \qquad \dim x_i : odd,$$

where p is a fixed prime. Suppose that there are classes $y_i \in H^*(P_n(X); \mathbf{Z}/p)$ so that each y_i restricts to the suspension of x_i in $H^*(\Sigma X; \mathbf{Z}/p)$ by the homomorphism induced by the inclusion $\Sigma X \subset P_n(X)$. (This property is referred as the A_n-primitivity of x_i.) Then there is an ideal S in $H^*(P_n(X); \mathbf{Z}/p)$ closed under the action of the Steenrod operation, so that