HIROSHIMA MATH. J.
25 (1995), 519–525

Links with homotopically trivial complements are trivial

Kazushi KOMATSU
(Received May 20, 1994)

1. Introduction

A smooth (resp. PL locally flat or locally flat) m-component link L stands for m embedded disjoint n-spheres $L_1 \cup \cdots \cup L_m$ in S^{n+2}. A knot is nothing but a 1-component link. A smooth (resp. PL locally flat or locally flat) m-component link is called trivial if it bounds m smoothly (resp. PL locally flatly or locally flatly) embedded disjoint $(n+1)$-disks. The complement of a trivial knot has the homotopy type of a circle S^1. The converse is known to be true for a locally flat knot. The converse is also known to be true for a smooth (or PL locally flat) knot when $n \neq 2$ ([9] for $n \geq 4$, [14] for $n = 3$ and [13] for $n = 1$). The complement of a trivial m-component link has the homotopy type of a one point union $(\vee_{i=1}^{m} S_i^1) \vee (\vee_{j=1}^{m-1} S_j^{n+1})$ of circles and $(n+1)$-spheres. So, there arises a natural question whether a link is trivial if the complement of the link has the homotopy type of a one point union of spheres. One of the purposes of this paper is to settle this question affirmatively provided that $n \neq 2$:

THEOREM 1. *Let $L \subset S^{n+2}$ be a smooth (resp. PL locally flat or locally flat) m-component link such that $S^{n+2} - L$ has the homotopy type of $(\vee_m S^1) \vee (\vee_{m-1} S^{n+1})$. Suppose that $n \neq 2$. Then L is trivial.*

A one point union of spheres has a special property that it is covered by two subsets which are contractible. This property itself is not a homotopy type invariant and a better notion is that it has Lusternik-Schnirelmann category one. The category $\operatorname{cat} X$ of a space X is the least integer n such that X can be covered by $n+1$ number of open subsets each of which is contractible to a point in X. In particular, $\operatorname{cat} X$ is a homotopy type invariant and $\operatorname{cat}((\vee_m S^1) \vee (\vee_{m-1} S^{n+1})) = 1$. We know that $\pi_1(X)$ is a free group if X is a manifold and $\operatorname{cat} X = 1$ (cf. [5]).

A locally flat knot (S^{n+2}, S^n) is topologically unknotted if and only if the category of its complement is one [11]. In fact, $\operatorname{cat}(S^{n+2} - S^n) = 1$ if and only if $S^{n+2} - S^n$ has the homotopy type of S^1. So, a smooth (or PL locally flat) knot (S^{n+2}, S^n) is unknotted if and only if $\operatorname{cat}(S^{n+2} - S^n) = 1$ when $n \neq 2$.

By Theorem 1 of [8] the link complement $S^{n+2} - L$ has the homotopy