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The Cowling–Price theorem for semisimple Lie groups

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ABSTRACT. M. G. Cowling and J. F. Price showed a generalization of Hardy's theorem as follows. If v and w grow very rapidly, then the finiteness of $\|vf\|_p$ and $\|w\hat{f}\|_q$ implies that $f = 0$, where $\hat{f}$ denotes the Fourier transform of f. We give an analogue of this theorem for the Helgason–Fourier transform for homogeneous vector bundles over Riemannian symmetric spaces and for connected noncompact semisimple Lie groups with finite centre.

1. Introduction

The mathematical uncertainty principle, roughly speaking, states that a nonzero function and its Fourier transform cannot both be sharply localized. First of all, in the case of Euclidean space, G. H. Hardy showed that if a measurable function f on $\mathbf{R}$ satisfies $|f(x)| \leqq Ce^{-ax^2}$ and $|\hat{f}(y)| \leqq Ce^{-by^2}$ and $ab > 1/4$, then $f = 0$ (a.e.). Here we use the Fourier transform defined by $\hat{f}(y) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(x) e^{\sqrt{-1}xy}\,dx$. M. G. Cowling and J. F. Price [3] generalized Hardy's theorem as follows. Suppose that $1 \leqq p, q \leqq \infty$ and one of them is finite. If a measurable function f on $\mathbf{R}$ satisfies $\|\exp\{ax^2\} f(x)\|_{L^p(\mathbf{R})} < \infty$ and $\|\exp\{by^2\} \hat{f}(y)\|_{L^q(\mathbf{R})} < \infty$ and $ab \geqq 1/4$, then $f = 0$ (a.e.). The case where $p = q = \infty$ and $ab > 1/4$ is covered by Hardy's theorem. S. C. Bagchi and S. K. Ray [1] showed that if $ab > 1/4$, then Hardy's theorem on $\mathbf{R}$ is equivalent to the Cowling–Price theorem.

Some generalizations of Hardy's theorem and the Cowling–Price theorem to various homegeneous spaces were obtained (e.g. [1], [4], [5], [6] and [12]). In these papers, the theorems were proved by using the estimates of matrix elements of representations and the Phragmén–Lindelöf theorem. The purpose of this paper is to prove an analogue of the Cowling–Price theorem for semisimple Lie groups. On the other hand, J. Sengupta [11] proved the Cowling–Price theorem on Riemannian symmetric spaces, by using the argument that the Fourier transform is decomposed into the composition of the Radon transform and the Euclidean Fourier transform. We consider the Helgason–

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