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Minimal sets of certain annular homeomorphisms

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Abstract. We consider a homeomorphism of the annulus $S^1 \times \mathbf{R}$ of the form $F_{\alpha,\varphi}(x, y) = (x + \alpha, y + \varphi(x))$, where α is an irrational number and φ is a continuous function on S^1 with vanishing integral. We show that if φ is of bounded variation and if $F_{\alpha,\varphi}$ is not topologically conjugate to $F_{0,\varphi}$, then $F_{\alpha,\varphi}$ does not admit a minimal set. We also show the abundance of such homeomorphisms.

1. Introduction

In [I], T. Inaba constructed an example of a smooth flow without a minimal set on an open surface of infinite genus. This was generalized by J.-C. Beniere and G. Meigniez [BM] to show that there are always flows without minimal sets on any noncompact manifolds other than the real line and surfaces of finite genus. This made us interested in considering the same problem for homeomorphisms on open surfaces.

Let f be a homeomorphism of a metric space X. A subset $\mathcal{M}$ of X is called a *minimal set* if $\mathcal{M}$ is a nonempty closed subset invariant by the homeomorphism f, which is minimal among such subsets with respect to the inclusion. By Zorn's lemma any homeomorphism on a compact space admits a minimal set. However this is no longer the case for a noncompact space.

If a homeomorphism f does not admit a minimal set, then there is no discrete orbit, since such an orbit would be a minimal set. As a consequence either the α-limit set or the ω-limit set of any point is nonempty, and thus the nonwandering set of f is nonempty. It follows for example that any homeomorphism of the plane must have a minimal set, because any homeomorphism with nonempty nonwandering set has a fixed point by a classical result of L. E. J. Brouwer. See *e.g.* [G].

Therefore first example of surfaces to be considered is an open annulus. Here we deal only with a special type of homeomorphisms, called *skew products*, which are defined as follows. Denote by R_α the rotation by α of the

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