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Note on γ-Operations in KO-Theory

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§1. Introduction

Let $p_i(\alpha)$ be the i-th (integral) Pontrjagin class of a real stable vector bundle α over a finite CW-complex X, and let γ^i be the Grothendieck γ-operation in KO-theory. Let k be a positive integer. Consider the two conditions: $p_k(\alpha)=0$ and $\gamma^{2k}(\alpha)=0$.

M. F. Atiyah has shown the following result in [3, §6] using the Chern character.

THEOREM 1.1. (M. F. Atiyah) *Suppose that $H^*(X; Z)$ is free. Then, for any real stable vector bundle α over X and for any positive integer k,*

$$\gamma^{2k}(\alpha) = 0 \Rightarrow p_k(\alpha) = 0\,.$$

For integers $n>0$ and $q>1$, we denote by $L^n(q)(=S^{2n+1}/Z_q)$ the $(2n+1)$-dimensional standard lens space mod q and by $RP^n(=S^n/Z_2)$ the real projective n-space. The purpose of this note is to prove the following

THEOREM 1.2. (i) *Assume that q is an odd integer>1. Let α be any real stable vector bundle over $L^n(q)$ and k be any positive integer. Then*

$$\gamma^{2k}(\alpha) = 0 \Rightarrow p_k(\alpha) = 0\,,$$

while the converse does not hold in general.

(ii) *The same is true for RP^n.*

There are examples of vector bundles for which the equality $\gamma^{2k}(\alpha)=0$ does not imply the equality $p_k(\alpha)=0$. Let CP^n $(=S^{2n+1}/S^1)$ be the complex projective n-space, and $D(m, n)$ be the Dold manifold of dimension $m+2n$ obtained from $S^m\times CP^n$ by identifying (x, z) with $(-x, \bar{z})$, where $(x, z)\in S^m\times CP^n$.

THEOREM 1.3. *Assume that $n=2^r$ and $m=2^s$ $(r>s>1)$. Let $\tau_0=\tau-(m+2n)$ be the stable class of the tangent bundle τ of $D(m, n)$, and put $k=n/2+m/4$. Then $\gamma^{2i}(-\tau_0)=0$ for any $i\geqq k$, but $p_k(-\tau_0)\neq 0$.*

Let η be the canonical complex line bundle over $L^n(q)$. In §2, we calculate the Pontrjagin class of a real stable vector bundle $\alpha=r\sum_{i=1}^{q-1}a_i(\eta^i-1)$, where