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The pseudo-convergent sets and the cuts of an ordered field

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Let F be an ordered field. A pair (A, B) of subsets of F is called a cut of F if $A \cup B = F$ and $a < b$ for any $a \in A$ and $b \in B$. In this paper we define the breadth of a cut of F which, in some sense, gives a measure of the gap between the lower class and the upper class.

The notion of pseudo-convergence with respect to the finest valuation among all compatible valuations in F plays an important role. Namely we can build up intrinsic relations between the cuts and the pseudo-convergent sets of elements of F. The limit of a pseudo-convergent set is by no means unique and the totality of limits can be described by the breadth of the pseudo-convergent set. We can show that the breadth of a pseudo-convergent set coincides with the breadth of the corresponding cut. As an application we give the following theorem: F has no strongly proper cut (see Definition 1.7) if and only if $A_0/M_0 = R$ and (F, v) is maximal as a valued field, where v is the finest valuation and (A_0, M_0) its valuation ring (Theorem 3.7).

§1. The finest valuation and cuts

For an ordered field F, let v be the finest valuation of F. The valuation ring of v is $A_0 := A(F, \boldsymbol{Q}) = \{a \in F;\ |a| < b \text{ for some } b \in \boldsymbol{Q}\}$. The maximal ideal and the value group of v will be denoted by M_0 and G respectively. A pair (A, B) of subsets of F is called a cut of F if $F = A \cup B$ and $A < B$.

DEFINITION 1.1. For a cut (A, B) of F, we put $E(A, B) = \{e \in F;\ b - a > |e| \text{ for any } a \in A \text{ and } b \in B\}$ and we call it the *breadth* of the cut (A, B). If $A = \phi$ or $B = \phi$, then we put $E(A, B) = F$. The breadth $E(A, B)$ is a convex additive subgroup of F.

The breadth of a cut (A, B) is characterized by $E(A, B) = \{e \in F;\ a + |e| \in A \text{ for any } a \in A\}$ or $E(A, B) = \{e \in F;\ b - |e| \in B \text{ for any } b \in B\}$. It is clear that a cut (A, B) is archimedean (for the definition, see [2], Definition 1.1) if and only if the breadth of (A, B) is zero.

DEFINITION 1.2. For a convex subgroup D of F, we put $A_1(D) = F^- \setminus D$, $B_1(D) = F^+ \cup D$, $A_r(D) = F^- \cup D$, $B_r(D) = F^+ \setminus D$, where F^+ (resp. F^-) is the set of positive (resp. negative) elements of F. Clearly $(A_1(D), B_1(D))$ and $(A_r(D), B_r(D))$ are cuts of