Nonoscillatory solutions of neutral differential equations

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(Received April 7, 1989)

1. Introduction

In this paper we are concerned with neutral differential equations of the form

\[ \frac{d^n}{dt^n} \left[ x(t) - h(t)x(\tau(t)) \right] + \sigma p(t)f(x(g(t))) = 0, \]

where \( n \geq 2 \), \( \sigma = 1 \) or \( -1 \), and the following conditions are always assumed to hold:

1. \( \tau(t) \in C[a, \infty) \), \( \tau \) is nondecreasing on \( [a, \infty) \), \( \tau(t) < t \) for \( t \geq a \) and \( \lim_{t \to \infty} \tau(t) = \infty \);
2. \( h(t) \in C[\tau(a), \infty) \), \( |h(t)| \leq h < 1 \) for \( t \geq a \), where \( h \) is a constant, and \( h(t)h(\tau(t)) \geq 0 \) for \( t \geq a \);
3. \( p(t) \in C[a, \infty) \) and \( p(t) > 0 \) for \( t \geq a \);
4. \( f(u) \in C((-\infty, \infty) \setminus \{0\}) \) and \( f(u)u > 0 \) for \( u \neq 0 \);
5. \( g(t) \in C[a, \infty) \) and \( \lim_{t \to \infty} g(t) = \infty \).

By a solution of (1.1) we mean a continuous function \( x \) which is defined and satisfies (1.1) on \( [T_x, \infty) \) for some \( T_x \geq a \) (so that \( x(t) - h(t)x(\tau(t)) \) is \( n \)-times continuously differentiable on \( [T_x, \infty) \)). Such a solution is said to be nonoscillatory if it has no zeros on \( [T, \infty) \) for some \( T \geq T_x \).

Recently there has been an increasing interest in the study of neutral differential equations, and a number of results have been obtained. For typical results we refer in particular to the papers [1–9, 14–18]. In this paper we make an attempt to study in a systematic way the structure of the set of nonoscillatory solutions of equation (1.1). In Section 2 we discuss the relation between two functions \( x(t) \) and \( x(t) - h(t)x(\tau(t)) \). The results obtained in Section 2 will be effectively used in subsequent sections. In Section 3 we classify the nonoscillatory solutions of (1.1) into several classes according to the asymptotic behavior as \( t \to \infty \). In Sections 4 and 5 we establish necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1) with specific asymptotic properties as \( t \to \infty \).