Existence theorems for a neutral functional
differential equation whose leading part contains
a difference operator of higher degree

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0. Introduction

In this paper we are concerned with the problem of existence of solutions
for a neutral functional differential equation of the form

\[(A) \quad D^n \Delta^m \lambda x(t) + f(t, x(g(t))) = 0, \quad t \geq t_0,\]

where \(D^n\) and \(\Delta^m\) stand, respectively, for the \(n\)-th iterate of the differential
operator \(D\) and the \(m\)-th iterate of the difference operator \(\Delta\) defined by

\[(0.1) \quad Dx(t) = \frac{d}{dt} x(t) \quad \text{and} \quad \Delta \lambda x(t) = x(t) - \lambda x(t - \tau).\]

In case \(\lambda = 1\) use is made of the symbol \(\Delta\) instead of \(\Delta_1\), i.e.,

\[(0.2) \quad \Delta x(t) = x(t) - x(t - \tau).\]

The conditions always assumed for (A) are as follows:

\[(0.3) \quad (a) \quad m \geq 1, \quad n \geq 1, \quad \lambda > 0, \quad \tau > 0 \quad \text{and} \quad t_0 > 0;\]

\[(b) \quad g \in C[t_0, \infty), \quad \text{and} \quad \lim_{t \to \infty} g(t) = \infty;\]

\[(c) \quad f \in C([t_0, \infty) \times \mathbb{R}), \quad \text{and} \quad |f(t, x)| \leq F(t, |x|), \quad (t, x) \in [t_0, \infty) \times \mathbb{R},\]

for some continuous function \(F(t, u)\) on \([t_0, \infty) \times \mathbb{R}_+\), \(\mathbb{R}_+ = [0, \infty)\),

which is nondecreasing in \(u\) for each fixed \(t \geq t_0\).

By a solution of (A) we mean a function \(x \in C[T_x - m\tau, \infty)\) for some
\(T_x \geq t_0 + m\tau\) such that \(D^n \Delta^m \lambda x(t)\) is \(n\)-times continuously differentiable and satisfies
the equation on \([T_x, \infty)\). A solution of (A) is said to be oscillatory if it has
an infinite sequence of zeros clustering at \(t = \infty\); otherwise a solution is said
to be nonoscillatory.

We observe that the associated unperturbed equation \(D^n \Delta^m \lambda x(t) = 0\) has
the solutions