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SCHWARZ'S LEMMA IN H_p SPACES

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§ 1. Introduction.

Let R be a Riemann surface and let $t\in R$ be any fixed point. For $0<p<\infty$, let $H_p(R)$ denote the class of all functions f analytic on R for which the subharmonic function $|f|^p$ has a harmonic majorant. We put for any $f\in H_p(R)$

$$\|f\|_p=(u(t))^{1/p}, \tag{1}$$

where u is the least harmonic majorant of $|f|^p$ on R. Then, for $1\leqq p<\infty$, $H_p(R)$ is a Banach space with the norm $\|\ \|_p$, and for $0<p<1$, $H_p(R)$ is not a Banach space but a Fréchet space with the metric $d(\ ,\ )$ defined by $d(f, g)=\|f-g\|_p^p$ ($f, g\in H_p(R)$). Although the "norm" $\|\ \|_p$ defined by (1) depends on the choice of t, the induced topology does not ([11]). Let $H_\infty(R)$ be the Banach algebra of all functions which are analytic and bounded on R, with the uniform norm $\|\ \|_\infty$. These H_p spaces, which generalize the classical Hardy classes in the unit disc, were introduced by Parreau [10] and independently by Rudin [11].

In this paper we are concerned with the problem of maximizing $|f'(t)|$ under the restrictions $f\in H_p(R)$, $f(t)=0$ and $\|f\|_p\leqq 1$. Let H_p^0 denote the class which consists of all $f\in H_p(R)$ such that $f(t)=0$ and $\|f\|_p\leqq 1$. We put for $0<p\leqq\infty$

$$\alpha_p=\sup_{f\in H_p^0}|f'(t)|.$$

We shall investigate some properties of α_p as a function of p on $(0, \infty]$. It is easily shown by the normal family argument that there exists a function $f\in H_p^0$ for which $f'(t)=\alpha_p$. Such a function is called an extremal function for H_p^0 and denoted by f_p. If $1<p<\infty$, then the uniform convexity of $H_p(R)$ implies that f_p is unique for any Riemann surface. It is well known that for any plane region there is a unique extremal function f_∞ for H_∞^0 ([5]). In this paper we shall also investigate the convergence of f_p as p approaches to some p_0 with $1<p_0\leqq\infty$. In Section 5, we shall consider another extremal problem similar to the above one.

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