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SOME EXTREMAL PROPERTIES IN THE UNIT BALL OF VON NEUMANN ALGEBRAS

BY HISASHI CHODA, YŌICHI KIJIMA AND YOSHIOMI NAKAGAMI

This paper is prepared to investigate some extremal properties in the unit ball of von Neumann algebras. Throughout this paper, by extremal point we mean the extremal point of the unit ball of the algebra considered. Theorem 1 is characterizations of extremal points. Theorems 2 and 6 are characterizations of finite von Neumann algebras. Theorem 3 gives a sufficient condition for a von Neumann algebra to be finite. Theorem 4 treats the extremal points of a von Neumann algebra which is induced into or reduced to the invariant subspace of the algebra or its commutant. Theorem 5 gives a necessary and sufficient condition for a von Neumann algebra to be a properly infinite factor. Theorems 6 and 7 treat the extremal points which appear in the tensor products. Theorems 1 and 2 are specializations of the results obtained by Kadison [2], Sakai [4], and Miles [3].

1. Notations and definitions. Let $\mathfrak{H}$ be a complex Hilbert space and $\mathfrak{L}(\mathfrak{H})$ be the full operator algebra on it. Let $\mathfrak{A}$ and $\mathfrak{B}$ be von Neumann algebras, and $\boldsymbol{C}$ the von Neumann algebra of all scalar multiples of the identity operator. For a projection E in $\mathfrak{A}$ or $\mathfrak{A}'$ the set $\{T_E : T\in\mathfrak{A}\}$ forms a von Neumann algebra $\mathfrak{A}_E$, where T_E is a restriction of ET to the range of E. For convenience, we shall denote by $\mathfrak{A}_1$ the unit ball of $\mathfrak{A}$, $\mathfrak{A}^e$ the set of extremal elements of $\mathfrak{A}_1$, $\mathfrak{A}^p$ the set of projections in $\mathfrak{A}$ and $\mathfrak{A}^{pi}$ the set of partially isometric operators in $\mathfrak{A}$. The operators 1 and 1_G stand for the identity of $\mathfrak{A}$ and $\mathfrak{A}_G$, where G is a projection belonging to the center of $\mathfrak{A}$. Furthermore, denote by $\mathfrak{A}^i$ the set of isometric operators in $\mathfrak{A}$, by $\mathfrak{A}^{i*}$ the set of A with $A^*\in\mathfrak{A}^i$ and $\mathfrak{A}^u$ the set of unitary operators in $\mathfrak{A}$. For E and $F\in\mathfrak{A}^p$, $E\sim F$ if and only if there is $A\in\mathfrak{A}$ with $A^*A=E$ and $AA^*=F$, and $E\prec F$ if and only if there is $A\in\mathfrak{A}$ with $A^*A=E$ and $AA^*\leqq F$. Let $\mathrm{Re}\,(x, y)$ be the real part of the inner product (x, y) for vectors x and y. Let $\mathfrak{A}\times\mathfrak{B}$ be the product von Neumann algebra of $\mathfrak{A}$ and $\mathfrak{B}$, and $A\times B$ be the product operator in $\mathfrak{A}\times\mathfrak{B}$ with $A\in\mathfrak{A}$ and $B\in\mathfrak{B}$. Let $\mathfrak{A}\otimes\mathfrak{B}$ be the tensor product of $\mathfrak{A}$ and $\mathfrak{B}$, and $\mathfrak{A}^e\otimes\mathfrak{B}^e$ denotes the set of tensor products $A\otimes B$ of all the pairs $A\in\mathfrak{A}^e$ and $B\in\mathfrak{B}^e$.

2. The following theorem due to Kadison plays an important role in this paper and the independent proof will be given.

THEOREM 1. *The following conditions*[1] *are equivalent*:

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1) Kadison has proved the mutual equivalence of (1) and (3) for C^*-algebra [2].