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ON THE SOLUTION OF THE FUNCTIONAL EQUATION $f\circ g(z)=F(z)$, III

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In our previous papers [3], [4] we discussed transcendental entire solutions of the functional equation $f\circ g(z)=F(z)$ and gave several transcendental unsolvability criteria, which based upon the existence of a Picard exceptional value, perfectly branched values, finite asymptotic paths and so on. All the criteria proved there do not work when F is an entire function of order less than 1/2 and even when $F(z)$ is $1/\Gamma(z)$. In this note we shall give a very useful criterion, which is based upon an elegant theorem due to Edrei [2] and which does work to some entire functions of order less than 1/2 and to $1/\Gamma(z)$ and the n-th Bessel function $J_n(z)$. And we shall give certain variants of this result. Further we shall give several criteria based upon Denjoy-Carleman-Ahlfors theorem.

Let $f(z)$ be an entire function and $M_f(r)$ its maximum modulus on $|z|=r$. We shall use the following notations:

$$\rho_f=\varlimsup_{r\to\infty}\frac{\log\log M_f(r)}{\log r},\qquad \lambda_f=\varliminf_{r\to\infty}\frac{\log\log M_f(r)}{\log r}$$

and

$$\hat{\rho}_f=\varlimsup_{r\to\infty}\frac{\log\log\log M_f(r)}{\log r},\qquad \hat{\lambda}_f=\varliminf_{r\to\infty}\frac{\log\log\log M_f(r)}{\log r}.$$

LEMMA 1. [4]. *$\rho_f<\infty$ implies $\hat{\rho}_{f\circ g}\leqq\rho_g$.*

LEMMA 2. *$\lambda_f>0$ implies $\hat{\rho}_{f\circ g}\geqq\rho_g$ and $\hat{\lambda}_{f\circ g}\geqq\lambda_g$.*

Proof. By Pólya's method we have

$$M_{f\circ g}(r)\geqq M_f\circ\left(d\,M_g\left(\frac{r}{2}\right)\right)$$

for a constant d, $0<d<1$. For a sufficiently small positive number ε there is an r_0 such that for $r\geqq r_0$

$$\log\log M_f(r)>(\lambda_f-\varepsilon)\log r$$

and there is a sequence $\{r_n\}$ of radii such that

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