ON COEFFICIENT PROBLEMS FOR SOME PARTICULAR CLASSES OF ANALYTIC FUNCTIONS

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1. Introduction.

Recently, Hummel [3] has established a variational formula on the class of univalent functions starlike in the unit circle. The variation employed consists in displacing every boundary point of the image domain in the radial direction with respect to the center of starlikeness. He then has applied the variational formula thus obtained to a coefficient problem of a general nature for the class under consideration. His result states:

Let $F(a_2, a_3, \cdots, a_n)$ be any function of the $n-1$ complex variables a_2, $a_3, \cdots, a_n$ having a continuous derivative in each variable. Then any function $f(z)=z+\sum_{\nu=2}^{\infty} a_\nu z^\nu$ which maximizes $\Re F(a_2, a_3, \cdots, a_n)$ within the class starlike in $|z|<1$ with respect to the origin must be of the form

$$f^*(z)=z\prod_{\mu=1}^{m}(1-\kappa_\mu z)^{-\sigma_\mu}$$

where $|\kappa_\mu|=1$, $\sigma_\mu>0$ for all μ $(1\leqq\mu\leqq m)$, $\sum_{\mu=1}^{m}\sigma_\mu=2$, and $m\leqq n-1$.

Hummel's variational formula is itself of considerable interest. His procedure of obtaining the above-mentioned result would seem also an interesting attempt to apply the variational method. The proof described in his paper depends on a linearization process, namely on the reduction of the problem of maximizing $\Re F(a_2, a_3, \cdots, a_n)$ to that of maximizing $\Re\sum_{\nu=2}^{n}\lambda_\nu a_\nu$, λ_ν being the value of $\partial F/\partial a_\nu$ corresponding to a maximizing function. Accordingly, it must be indeed supposed, even if not explicitly stated, that at least one among the λ's does not vanish, what seems, however, a priori not self-evident. Nevertheless, Hummel's result remains valid. It will be further shown that the result can be derived more simply without making use of the variational formula. Namely, we shall give in the present paper a new proof of the Hummel's theorem. For that purpose, we consider previously an analogous coefficient problem for a related class of functions.

2. The class $\Re$.

Let $\Re$ be the class which consists of analytic functions $\Phi(z)$ with positive real part in $|z|<1$ and normalized by $\Phi(0)=1$. Let the Taylor expansion of $\Phi(z)$ be

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