EXISTENCE OF GREATEST DECOMPOSITION OF A SEMIGROUP

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§ 1. Let us consider letters x_1, x_2, ..., x_n placed in a row permitting repetition, for example, $x_2 x_1^2 x_3^2 x_n x_1 \cdots x_{n-1}^4$. Such a form is called a monomial of x_1, ..., x_n and is denoted by $f(x_1, \cdots, x_n)$ etc. If x_1, ..., x_n are adopted as elements of a semigroup T, then $f(x_1, \cdots, x_n)$ represents a product of elements in T. Suppose that a semigroup T fulfils a suitable system of equalities:

$$f_\lambda(x_1, \cdots, x_n) = g_\lambda(x_1, \cdots, x_n), \quad \lambda \in \Lambda,$$

for all $x_1, \cdots, x_n \in T$,

where x_1, ..., x_n vary independently and each side of the equalities needs not contain all of letters x_1, ..., x_n, but letters appearing in both sides are x_1, ..., x_n; for example, $f_1(x, y) = xy$, $g_1(x, y) = x$. Then T is called a semigroup with monomial conditions $f_\lambda = g_\lambda$, $\lambda \in \Lambda$. Of course a one-element semigroup $\{x\}$ is one of this kind.
In this short note, we shall prove the existence of greatest decomposition of a semigroup S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$, which turns out to be an expansion of the theorem in the previous paper [1].

§ 2. Now let D be all decompositions of S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$, and $\overset{d}{\sim}$ be a congruence relation arising $d \in D$. The following lemma is clear.

Lemma 1. $\overset{d}{\sim}$ is a congruence relation arising a decomposition d of S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$, if and only if

(1) $x \overset{d}{\sim} x$,
(2) $x \overset{d}{\sim} y$ implies $y \overset{d}{\sim} x$,
(3) $x \overset{d}{\sim} y$ implies $xz \overset{d}{\sim} yz$ and $zx \overset{d}{\sim} zy$,
(4) $f_\lambda(x_1, x_2, \cdots, x_n) \overset{d}{\sim} g_\lambda(x_1, x_2, \cdots, x_n)$, $\lambda \in \Lambda$.

Theorem. D is a complete lattice.

Proof. We define $d_\alpha \geqq d_\beta$ to mean that $x \overset{d_\alpha}{\sim} y$ implies $x \overset{d_\beta}{\sim} y$.
Then D is a partly ordered set and D contains a least element, i.e. a mapping of all elements of S into one class. In order to verify that D is a complete lattice, it is sufficient to show that any subset D' of D has a least upper bound in D [2]. Now we define $x \overset{d_1}{\sim} y$ to mean $x \overset{d}{\sim} y$ for all $d \in D'$. Since every $\overset{d}{\sim}$ is a congruence relation, it is proved easily that $\overset{d_1}{\sim}$ is also so, that is, (1) $x \overset{d_1}{\sim} x$, (2) $x \overset{d_1}{\sim} y$ implies $y \overset{d_1}{\sim} x$, (3) $x \overset{d_1}{\sim} y$ implies $xz \overset{d_1}{\sim} yz$ and $zx \overset{d_1}{\sim} zy$.
Moreover $f_\lambda(x_1, x_2, \cdots, x_n) \overset{d_1}{\sim} g_\lambda(x_1, x_2, \cdots, x_n)$ because $f_\lambda(x_1, x_2, \cdots, x_n) \overset{d}{\sim} g_\lambda(x_1, x_2, \cdots, x_n)$ for all $d \in D'$. Obviously $x \overset{d_1}{\sim} y$ implies $x \overset{d}{\sim} y$ for all $d \in D'$; hence a decomposition d_1 is an upper bound of D'.
Let d_1' be any upper bound of D'.
Then $x \overset{d_1'}{\sim} y$ implies $x \overset{d}{\sim} y$ for all $d \in D'$ so that $x \overset{d_1}{\sim} y$, that is to say, $d_1' \geqq d_1$; d_1 is a least upper bound of D'.
Thus the proof of the theorem has been completed. Accordingly we have

Corollary. There is a greatest decomposition of a semigroup S to a semigroup T with $f_\lambda = g_\lambda$, $\lambda \in \Lambda$.

§ 3. We shall give several important examples of T.

1. Left singular semigroup, i.e. a semigroup satisfying $xy = x$,

$$f_1(x, y) = xy, \quad g_1(x, y) = x.$$

Right singular semigroup, i.e. a semigroup satisfying ,

$$f_1(x, y) = xy, \quad g_1(x, y) = y.$$

2. Commutative semigroup,

$$f_1(x, y) = xy, \quad g_1(x, y) = yx.$$