ON TRANSLATIONS OF A SEMIGROUP

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Let S be a semigroup. We mean by a right translation of S a mapping φ of S into S such that

$$\varphi(xy) = x\varphi(y)$$

for every $x, y \in S$.

Analogously we define a left translation ψ as

$$\psi(xy) = \psi(x)y$$

for every $x, y \in S$.

The set of all right translations of forms a semigroup by usual product defined as $(\varphi_\beta \varphi_\alpha)(x) = \varphi_\beta(\varphi_\alpha(x))$; and the set is denoted by Φ which is called a right translation semigroup of S. Similarly a left translation semigroup Ψ of S is defined. If S is a commutative semigroup, the distinction between "right" and "left" is not required.

In this paper we shall arrange fundamental properties of a translation semigroup and shall mention that some special types of semigroups are characterized by their translation semigroups. We may omit dual proof for simplicity, namely, prove propositions only in the "right" case and do not in the "left"case.

Now we denote by M the set of all mappings of S into S. Naturally M is a semigroup and $\Phi \subset M$, $\Psi \subset M$.

Lemma 1. Φ and Ψ are semigroups with two-sided unit.

Proof. If $\varphi_1, \varphi_2 \in \Phi$, then it holds that

$$(\varphi_2\varphi_1)(xy) = \varphi_2(\varphi_1(xy))$$

$$= \varphi_2(x\varphi_1(y)) = x\varphi_2\varphi_1(y)$$

for every $x, y \in S$. Hence $\varphi_1\varphi_2 \in \Phi$. Whatever S is, an identical mapping φ_0 of S to S is included in Φ , and φ_0 is obviously a two-sided unit of Φ since

$$\varphi_0\varphi = \varphi\varphi_0 = \varphi$$

for every $\varphi \in \Phi$.

Let f_a be a mapping of S into S which is given as

$$f_a(x) = xa \qquad a \in S ,$$

and let g_a be a mapping defined as $g_a(x) = ax$. This f_a is called an inner right translation, and g_a is called an inner left translation. Denote by R the set of all f_a where a flows throughout S , and by L the set of all g_a, $a \in S$. Obviously

$$R \subset \Phi, \quad L \subset \Psi.$$

As well-known, S is homomorphic to L, and dually homomorphic to R.

Lemma 2. R is a right ideal of S and L is a left ideal of S.

Proof. For every $\varphi \in \Phi$ and $f_a \in R$,

$$(\varphi f_a)(x) = \varphi(f_a(x))$$

$$= \varphi(xa) = x\varphi(a) = f_{\varphi(a)}(x)$$

so that we get $\Phi R \subset R$.

Corollary. R and L are subsemigroups of Φ and Ψ respectively.

We have easily

Theorem 1. A mapping φ is a right